Quantum measurement and control in optomechanical systems
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Exercises of the first lecture

1. **The damped quantum harmonic oscillator.**

   The mean energy is determined by

$$\begin{aligned}\langle a^\dagger(t)a(t)\rangle &= \langle a^\dagger(0)a(0)\rangle e^{-\gamma t} \\ &\quad +\gamma e^{-\gamma t}\int_0^t dt_1\int_0^t dt_2 e^{-i(\omega_0+i\gamma/2)t_1}e^{i(\omega_0-i\gamma/2)t_2}\langle a_{in}^\dagger(t_1)a_{in}(t_2)\rangle\end{aligned}$$

   where $\langle a_{in}^\dagger(t_1)a_{in}(t_2)\rangle = N\delta(t_1-t_2)$.

   Using the Stochastic calculus rule:

$$\int_0^{t_1} dt'\int_0^{t_2} dt'' f^*(t')f(t'')\langle a_{in}^\dagger(t')a_{in}(t'')\rangle = \int_0^{min(t_1,t_2)} dt'|f(t)|^2\ N$$

   show that

$$\langle a^\dagger(t)a(t)\rangle = \langle a^\dagger(0)a(0)\rangle e^{-\gamma t} + N(1-e^{-\gamma t})$$

   In the steady state, $\langle a^\dagger(t)a(t)\rangle \to N$ ... thermal equilibrium.

2. **The driven damped quantum harmonic oscillator.**

   Show that in the steady state,

$$\langle a\rangle_{ss} = \frac{-\epsilon}{i\delta+\gamma/2}.$$

   Calculate the mean displacement and momentum in the steady state in the original laboratory frame.

3. **The driven damped quantum harmonic oscillator.**

   The noise power spectrum for the driving force is defined as the Fourier transform of the two-time correlation function $G(\tau)$:

$$S_f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau e^{-i\omega\tau}G(\tau)$$

   Show that for

$$G(\tau) = De^{-\kappa|\tau|}$$

   the noise power spectrum is Lorentzian.

4. **The driven damped quantum harmonic oscillator.**

The steady state response to the force can be most easily calculated in the frequency domain:

$$\tilde{a}(\omega) = \frac{\sqrt{\gamma}\tilde{a}_{in}(\omega) + \tilde{\epsilon}(\omega)}{\gamma/2 + i(\delta - \omega)}$$

where $\tilde{\epsilon}(\omega)$ is the Fourier transform of $\epsilon(t)$.

Assume that $\epsilon_0 = 0$, so that the oscillator is subject to a classical fluctuating force. Show that the steady state photon number is then determined by

$$\langle \tilde{a}^\dagger(\omega)\tilde{a}(\omega)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{\gamma N + 2\pi S_f(\omega)}{\gamma^2/4 + (\delta - \omega)^2}.$$

5. **Oscillator displacement.**

Define

$$G_x(\tau) = \langle x(t)x(t+\tau)\rangle_{t\to\infty}$$

and

$$S_x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} G_x(\tau)$$

Show that

$$S_x(\omega) = \frac{\gamma(2N+1)\Delta_0^2}{\gamma^2/4 + (\omega_0 - \omega)^2}.$$

On resonance and for zero temperature, this result gives the standard quantum limit (SQL)

$$S_x^{SQL} = \frac{2\hbar}{m\omega_0\gamma}.$$