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Husimi Distributions of a Spin–Boson System and the Signatures of Its Classical Dynamics

K. FURUYA* AND M. A. M. DE AGUIAR*

Instituto de Física "Gleb Wataghin," Departamento de Física do Estado Sólido e Ciências de Materiais, Universidade de Campinas, C.P. 6165, Campinas, Brazil

C. H. LEWENKOPF†

Max-Planck-Institut für Kernphysik, W-6900 Heidelberg, Germany

AND

M. C. NEMES*

Instituto de Física, Departamento de Física–Matemática, Universidade de São Paulo, C.P. 20516, São Paulo, Brazil

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We investigate the effect of the compact Hilbert space associated to the spin degree of freedom on the Husimi distributions of a spin-boson system (maser model). In the classical limit, the phase space associated to such a degree of freedom is limited to a circle of radius $\sqrt{4J}$ (J being the total spin). Therefore for high enough energies the classical trajectories exhibit many self crossings and cusps. In this work we show that such characteristic behavior is also reflected in the corresponding quantum regime. Marked intensity enhancements are found.

1. INTRODUCTION

In recent years much effort has been dedicated to the understanding of manifestations of chaos in quantum mechanics in "simple" Hamiltonian systems (two degrees of freedom), partially motivated by possible applications on more "complex" systems (many degrees of freedom) such as atoms, nuclei, and molecules. Beside the important question of the dimension of the phase space, there is another question which still deserves more investigation: The rôle played by the spin in many-body systems. Most of the known many-body systems are formed by particles with intrinsic spin. These particles couple to a total spin which is generically a good quantum

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number; its components, however, can exhibit a very interesting dynamics which correspondence with chaos would be very elucidative to understand. The difficulty starts on a very primary level: To put spin in correspondence with classical mechanics is not trivial because of the very structure of its Hilbert space. The main difference between a spin and particle degrees of freedom is in their associated Hilbert spaces: A finite Hilbert space is associated to the spin in distinction to the general situation where one faces an infinite Hilbert space. We will call a particle degree of freedom the one associated to an infinite Hilbert space. Although the $SU(3)$ Lipkin model has been extensively studied [1–4], not much more attention has been devoted to the consequences of such a difference. In this paper we present a study of a model Hamiltonian which has one spin degree of freedom and one particle degree of freedom. Moreover, this model has been shown to exhibit chaos as well as regular behaviour on dependence of some parameters. These facts make the model suitable for understanding the main differences arising from the finiteness of the phase space.

The first studies relating quantum mechanics and classical chaos were purely "phenomenological." They established a universal connection between the level statistics and the classical phase space empirically. Most of the efforts in order to establish a more fundamental connection between quantum and classical mechanics for nonintegrable systems are based on the semiclassical work of Gutzwiller [5]. Using the Gutzwiller trace formula and the Hannay–Ozório de Almeida sum rule [6], Berry [7] was able to give partial answers to the question of universality. The level statistics of the model considered in this paper was already extensively studied [8]. The results are in agreement with the empirical expected behaviour. Since the system is not scalable, a full understanding of the quantum spectrum in terms of the classical phase space is a more than ambitious project. On the other hand, recent investigations of the quantum wave functions of classical chaotic systems open novel interesting possibilities. Wave functions carry signatures of the classical phase space even for chaotic systems. Observation of the enhancement of eigenfunctions were reported in the numerical work of MacDonald and Kaufman [9] followed by Taylor and Brumer [10] for the quantum version of classical chaotic systems. Heller and collaborators [11] quoted the concentration of wave function densities around the region of periodic orbits and called them "scars." The corresponding analytical results were obtained by Bogomolny [12] and Berry [13]. In all the quoted works, systems associated to infinite Hilbert space have been considered. To our knowledge not as much effort has been devoted to finite Hilbert space systems as, for example, spin systems.

In a previous work [14], henceforth referred to as (I) a detailed numerical analysis of the classical limit of the maser model is given. There we show that one of the main effects of the finiteness of the spin phase space in the chaotic regime is the following: as the energy increases and due to the energetic border in the spin phase space the corresponding projections of periodic orbits (mainly unstable) exhibit many self crossings and cusps. In this contribution we show that such classical border effect has its quantum counterpart. We investigate the structure of exact individual wave functions via projections of Husimi distributions in both degrees of freedom. Most of them exhibit increases in intensity which correspond precisely to the vicinity of the classical periodic orbit self crossings or cusps. Moreover, in the particle projection of Husimi distributions we find scars of classical unstable periodic orbits (of smallest periods). When the contributing classical orbits are not isolated in period interference effects are present [15]. In the particle Husimi distributions the above-mentioned border effect is even more conspicuous: the intensity enhancement of the wave functions due to self crossings and cusps are very marked and survive even in the presence of interference effects.

The structure of the papers is as follows: In Section 2 we introduce the model. A discussion of the Husimi distribution as a tool for the understanding of the wave functions in terms of the classical phase space, as well as explicit formulas for the maser model are presented in Section 3. Section 4 contains a detailed numerical analysis of the Husimi distributions in both integrable and nonintegrable situations. Our conclusions are summarized in Section 5.

2. The Maser Model

The maser model was conceived in 1952 by Townes and realized experimentally bu Gordon, Zeiger, and Townes [16] three years later. The basic idea of the maser is the following: One works with ammonia molecules (NH_3), due to the fact that, despite its quite complicated structure, only two energy levels appear to be relevant for the maser. The nitrogen N can be located at either side of the hydrogen H plane with an energy splitting of approximately 12.500 MHz (corresponding to a wavelength greater than 1 cm). One can model this behaviour of the NH_3 molecule by a one-dimensional system.

Let us consider the nth atom of a two-level system and label its ground state by $|\theta_1^{(n)}\rangle$ and the state corresponding to level 2 by $|\theta_2^{(n)}\rangle$. Any operator acting in this system can be expanded in the set of Pauli matrices $\sigma_x^{(n)}, \sigma_y^{(n)}, \sigma_z^{(n)}$ plus the identity matrix $I^{(n)}$ associated with this particular atom. The two-level system discussed above is thus represented by a spin $\frac{1}{2}$ system for which spin-up and spin-down operators are defined by

$$\sigma_\pm^{(n)} = \tfrac{1}{2}(\sigma_x^{(n)} \pm \imath\sigma_y^{(n)}),$$

where the Pauli matrices obey the usual commutation rules

$$[\sigma_x^{(n)}, \sigma_\pm^{(n)}] = 2\sigma_\pm^{(n)}, \qquad [\sigma_+^{(n)}, \sigma_-^{(n)}] = \sigma_z^{(n)}.$$

The Hilbert space of the assembly of N atoms is spanned by the set of 2^N product states,

$$|\psi_{i_1 i_2 \cdots i_N}\rangle = \prod_{n=1}^{N} |\theta_{i_n}^{(n)}\rangle \qquad (i_n = 1, 2).$$

Collective angular momentum operators are now defined as

$$J_\mu = \frac{1}{2} \sum_n \sigma_\mu^{(n)} \qquad (\mu = x, y, z)$$

$$J_\pm = \sum_n \sigma_\pm^{(n)} \tag{1}$$

$$J^2 = J_x^2 + J_y^2 + J_z^2.$$

The coupling of this assembly of atoms with the electromagnetic field determines the dynamics of the system. The field operators, considering one single mode can be represented by Bose operators which are dynamically represented as a harmonic oscillator. The simplest Hamiltonian which realizes the Dicke maser model can now be written as [17, 18]

$$\hat{H} = \varepsilon_1 a^\dagger a + \varepsilon_2 J_z + \frac{G}{\sqrt{N}} (aJ_+ + a^\dagger J_-) + \frac{G'}{\sqrt{N}} (a^\dagger J_+ + aJ_-), \tag{2}$$

where $a^\dagger(a)$ are the Bose operators of the quantized mode with frequency ε_1. The spin operators represent at least $2J$ (J is the total spin) two level atoms with separation energy ε_2. In what follows we consider $\varepsilon_1 = \varepsilon_2 = 1$ and the coupling constants G and G' measured in units of ε.

An enormous amount of work has already been done in connection to the above Hamiltonian in the various field of its application. These can be traced back from references [17, 18].

3. Husimi Distributions

In this section we present the definition of Husimi distributions. It is well known that they constitute a powerful tool to investigate scars of periodic orbits on the wave functions [19, 20, 3, 4]. Essentially as described below Husimi distributions give the wave intensity in the coherent state representation. The appropriate coherent states for our model are the Glauber states (for the particle degree of freedom) and Bloch states (for the spin degree of freedom) [21–23].

In order to compare classical versus quantum features, we face the problem that the classical solution is best described as a phase-space trajectory, whereas the quantum one is described by a coordinate wave-function $\langle \mathbf{q} | \psi \rangle$ or momentum wave functions $\langle \mathbf{p} | \psi \rangle$ (with the Heisenberg's uncertainty principle forbidding the precise knowledge of $\mathbf{q}$ and $\mathbf{p}$ simultaneously).

It is known since Wigner [24] that one can define a distribution of both $\mathbf{q}$ and $\mathbf{p}$ as

$$W(\mathbf{q}, \mathbf{p}) = \frac{1}{(2\pi\hbar)^D} \left\langle \mathbf{q} + \frac{\mathbf{u}}{2} \middle| \psi \right\rangle \left\langle \psi \middle| \mathbf{q} - \frac{\mathbf{u}}{2} \right\rangle \exp\left(\frac{i\mathbf{p} \cdot \mathbf{u}}{\hbar}\right), \tag{3}$$

where D is the number of degrees of freedom. Though such a distibution function does provide by projection the correct probability densities,

$$\begin{aligned}\psi(\mathbf{q}) &= \int W(\mathbf{q}, \mathbf{p})\, d\mathbf{p} = |\langle \mathbf{q} | \psi \rangle|^2, \\ \psi(\mathbf{p}) &= \int W(\mathbf{q}, \mathbf{p})\, d\mathbf{q} = |\langle \mathbf{p} | \psi \rangle|^2,\end{aligned} \tag{4}$$

one is *not* allowed to interpret $W(\mathbf{q}, \mathbf{p})$ as a phase space probability density due to the inconvenience of it being necessarily negative in some regions of the phase space [25].

It has been conjectured that a smoothed version of the Wigner distribution, averaged over volumes larger than those of minimum uncertainty, would always provide a positive function. However, in a recent paper [26] a counterexample of such an averaging conjecture has been constructed showing that this may not always be true if the averaging is *not* Gaussian.

Hence, let us stick to the Gaussian smoothing of $W(\mathbf{q}, \mathbf{p})$ or the so-called *Husimi distribution* [27], which may be written as the wave intensity in the coherent state representation [28, 29]

$$h(z, w) \equiv |\langle z, w | \psi \rangle|^2, \tag{5}$$

where the coherent state $|z, w\rangle$ is given by the direct product of $|z\rangle = e^{-z\bar{z}/2} e^{za^\dagger} |0\rangle$ and $|w\rangle = (1 + w\bar{w})^{-J} e^{wJ_+} |J -J\rangle$, where z and w are complex numbers, $|0\rangle$ is the harmonic oscillator ground state, and $|J -J\rangle$ is the state with spin J with projection $J_z = -J$ on the z-axis.

One can interpret $h(z, w)$ as the probability of finding the system in a phase space region of volume $\hbar$ centered on the point (z, w).

The Husimi distributions are calculated for each eigenstate as follows: Let $|\psi_l\rangle$ be the lth eigenstate of the Hamiltonian operator $\hat{H}$ defined in Eq. (2). In terms of the basis $|n\rangle \otimes |Jm\rangle$, the direct product of a harmonic oscillator state times a state of spin J and projection $J_z = m$ on the z-axis, $|\psi_l\rangle$ is then explicitly given by

$$|\psi_l\rangle = \sum_{n=0}^{\infty} \sum_{m=-J}^{J} C^l_{nm} |n\rangle \otimes |Jm\rangle, \tag{6}$$

where the real coefficients C^l_{nm} have been calculated by solving the secular matrix for $\hat{H}$ under the chosen basis. Now, we project $|\psi_l\rangle$ into the coherent state $|z, w\rangle$ and take its modulus squared,

$$\begin{aligned}h_l(z, w) &\equiv |\langle z, w | \psi_l \rangle|^2 \\ &= \sum_{n,n'=0}^{\infty} \sum_{m,m'=-J}^{J} C^l_{n'm'} C^l_{nm} \langle w | Jm \rangle \langle Jm' | w \rangle \langle z | n \rangle \langle n' | z \rangle.\end{aligned} \tag{7}$$

Then, using the normalized version of the states $|z\rangle$ and $|w\rangle$, the functions $\langle n|z\rangle$ and $\langle Jm|w\rangle$ can be explicitly calculated,

$$\langle n|z\rangle = \frac{e^{-z\bar{z}/2} z^n}{\sqrt{n!}}$$
$$\langle Jm|w\rangle = \frac{w^{J+m}}{(1+w\bar{w})^J} \sqrt{\frac{(2J)!}{(J+m)!\,(J-m)!}}. \tag{8}$$

For systems with two degrees of freedom, like the Dicke Hamiltonian, $h_l(z, w)$ is a function of four real variables and, so, difficult to display. Therefore in order to visualize the behaviour of the particle and spin degrees of freedom we define the following projections of $h_l(z, w)$ in the space of particle and spin, respectively:

$$\begin{aligned} h_l(z) &= \int h_l(z, w)\, d[\mathrm{Re}(w)]\, d[\mathrm{Im}(w)]\, \frac{(2J+1)}{\pi(1+w\bar{w})^2} \\ &= \sum_{n,n'=0}^{\infty} \sum_{m=-J}^{J} C^l_{n'm} C^l_{nm} \frac{e^{-z\bar{z}} \bar{z}^{n'} z^n}{\sqrt{n'!}\,\sqrt{n!}} \end{aligned} \tag{9}$$

and

$$\begin{aligned} h_l(w) &= \int h_l(z, w)\, \frac{d[\mathrm{Re}(z)]\, d[\mathrm{Im}(z)]}{\pi} \\ &= \sum_{n=0}^{\infty} \sum_{m,m'=-J}^{J} C^l_{nm'} C^l_{nm} \frac{w^{2J+m+m'}}{(1+w\bar{w})^{2J}} \sqrt{\frac{(2J)!}{(J+m)!\,(J-m)!}} \\ &\quad \times \sqrt{\frac{(2J)!}{(J+m')!\,(J-m')!}}. \end{aligned} \tag{10}$$

Both $h_l(z)$ and $h_l(w)$ are now functions of only two real variables and their contour plots can indicate the regions where the probabilities associated with the particle and the spin are large. These plots can then be compared to the projection of classical structures presented in Paper I.

4. Numerical Results

In this section we compare the Husimi distributions as defined in the previous section to classical periodic orbits and provide for a detailed analysis of their signatures on the projected Husimi distributions for both degrees of freedom. The complete classical analysis of the present extension of the maser model has been given in (I). Therefore we shall briefly recall the main results, which are necessary for the comparison. As discussed in [30] the maser model presents a superradiant phase transition when the parameters are such that $(G+G') = \varepsilon$, this means, in par-

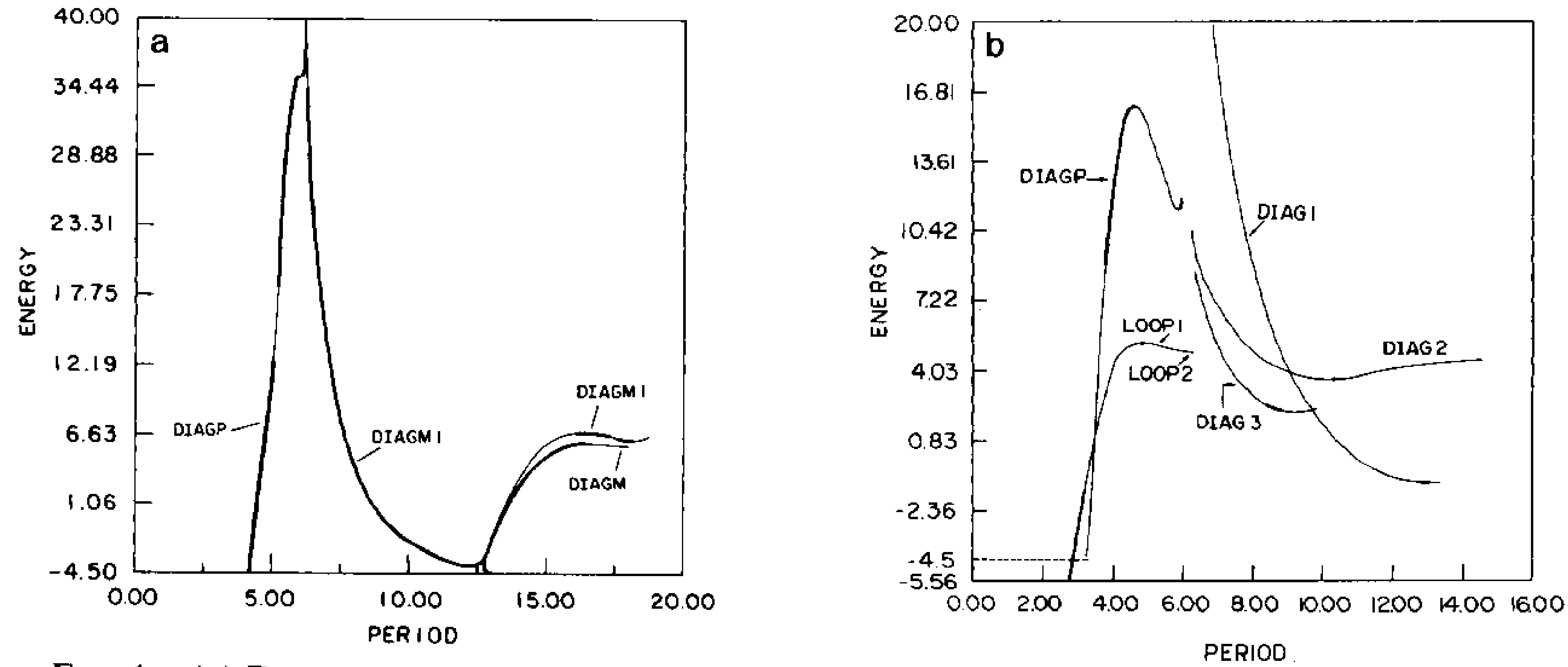

FIG. 1. (a) Energy versus period for the simplest families of periodic orbits. The coupling parameters are $G = 0.5$ and $G' = 0.2$. (b) Same as Fig. 1a for the coupling parameters $G = 1.0$ and $G' = 0.4$.

ticular, that a bifurcation of equilibria will occur. In order to visualize this situation and recall the corresponding families of periodic orbits (p.o.) we present the Energy $\times$ Period $(E \times \tau)$ plot for the two cases of interest:

(a) Before the phase transition with parameter values $G = 0.5$, $G' = 0.2$ (nonintegrable case, see Fig. 1a).

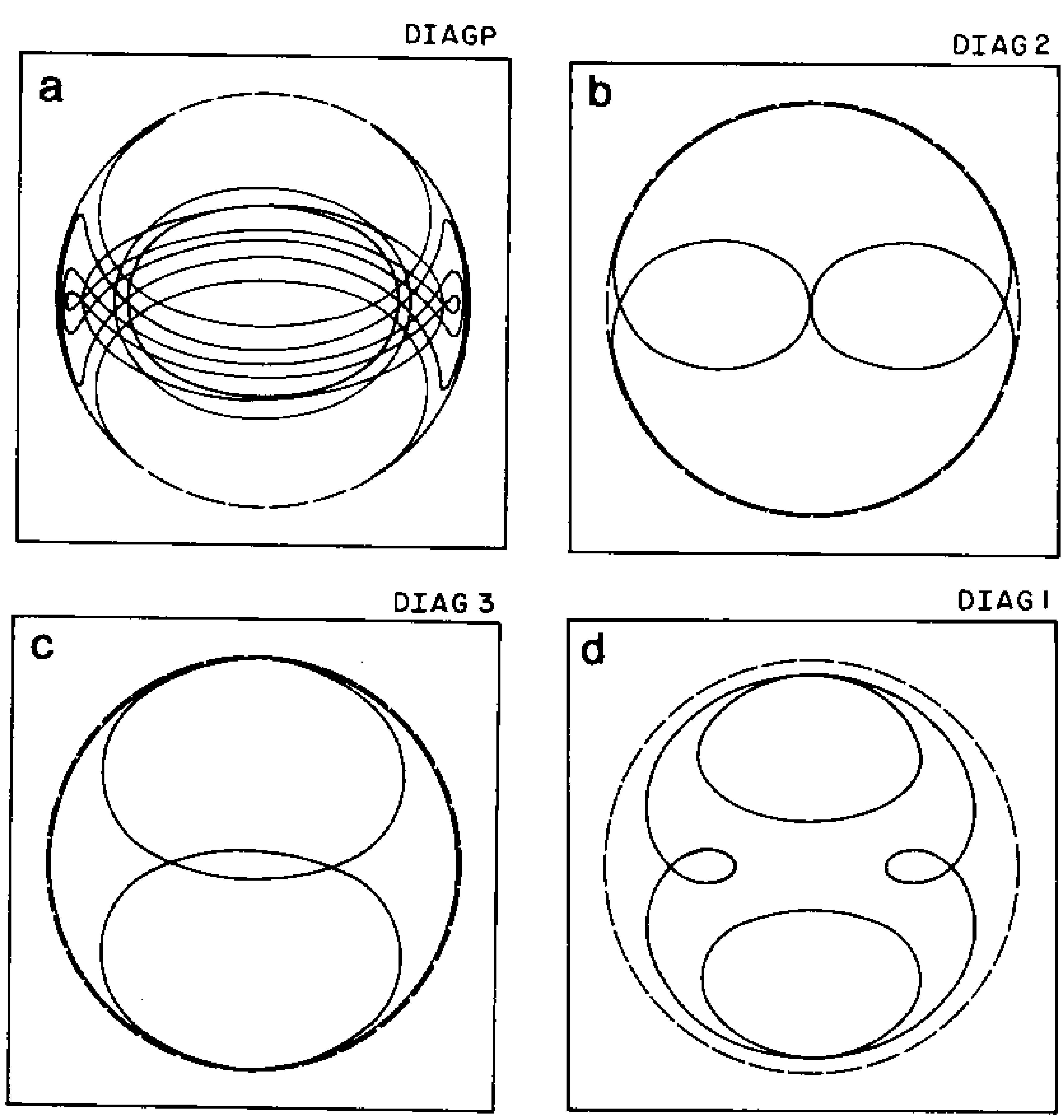

FIG. 2. (a) Sequence of spin projections of periodic orbits with parameter values as in Fig. 1b for the family DIAGP. (b)–(d) Spin projection of classical periodic orbits of the families indicated in the figures (parameter values of Fig. 1b).

(b) After the phase transition with parameter values $G=1$, $G'=0.4$ (see Fig. 1b).

The analysis of the families present in Figs. 1 as well as Poincaré sections are given in (I). Here we shall only refer to the family names and show the relevant trajectories for the sake of comparison with the quantal results.

We now proceed to examine the energy region around 8.5 corresponding to Fig. 1b. Note that in this energy region the orbit families have similar periods and could therefore interfere (see Fig. 1b). In Figs. 2(a–d) we show the spin projection of unstable classical p.o. belonging to the various families. It is important to observe the presence of many self-crossings and cusps due to the limitation in phase space, as discussed before. The spin projection of the Husimi distributions in this energy range exhibit intensity enhancements which can be attributed to the self crossings and cusps (compare, for example, Fig. 3a) with the trajectory in Fig. 2b, Figs. 3c and d seem to show enhancements at self crossings of more than one classical p.o., for instance, Figs. 2(a–b) (corresponding to 3c) and Figs. 2(c–d) (corresponding to Fig. 3d). Figure 3a should be compared to the trajectories in Fig. 2a. The intensity enhancement in this case seems to be connected to the presence of cusps in the classical p.o.

The corresponding particle projection of the Husimi distributions are shown in Figs. 4(a–d). Figure 4a shows a scar of the indicated p.o., DIAG2. The blackened

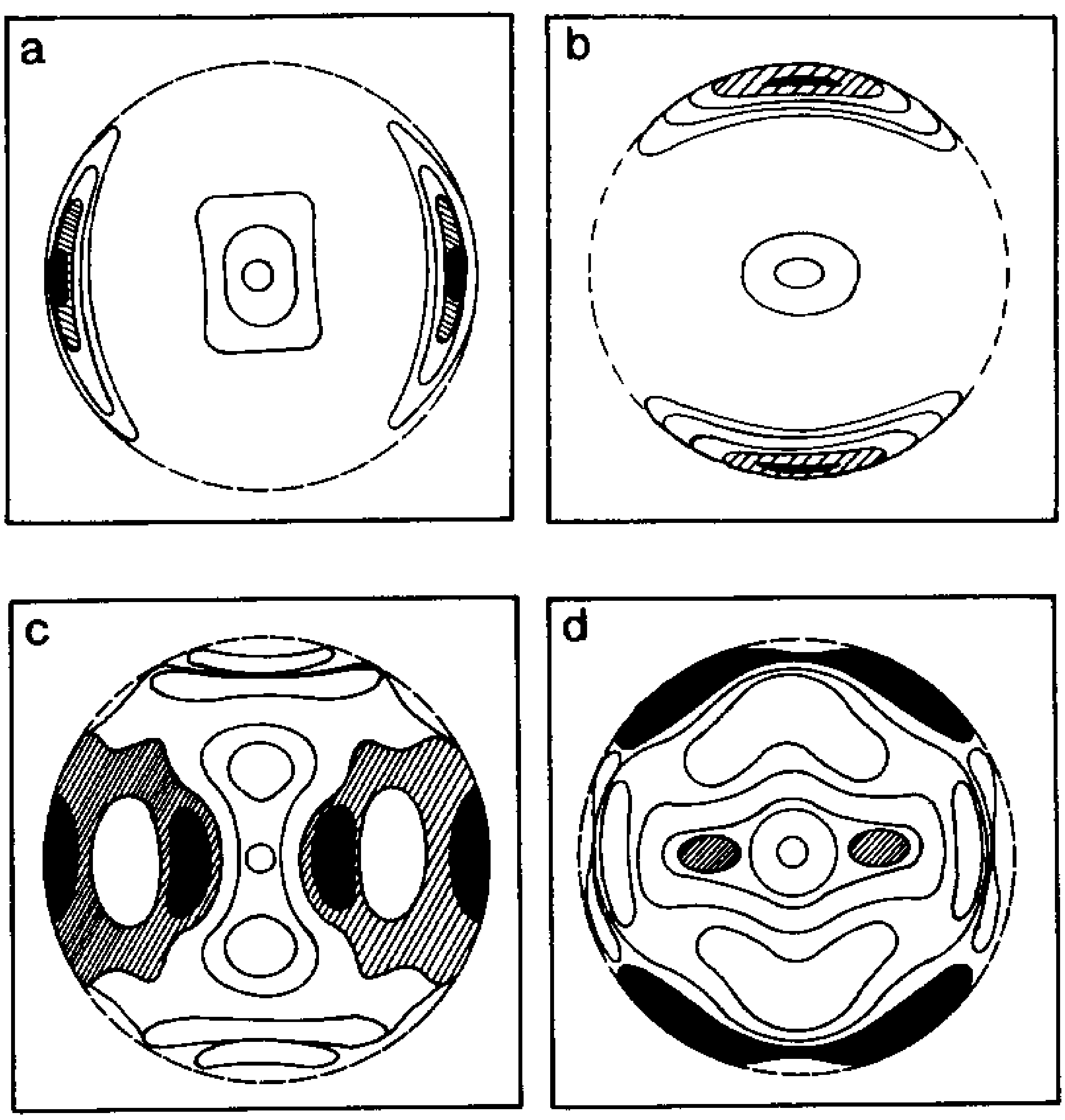

FIG. 3. (a)–(d) Contour plots of $h_l(w)$, where the blackened area represents the distribution from maximum to 95 % of the maximum and contours are drawn at 90, 80, and 70 % in figures with three contours.

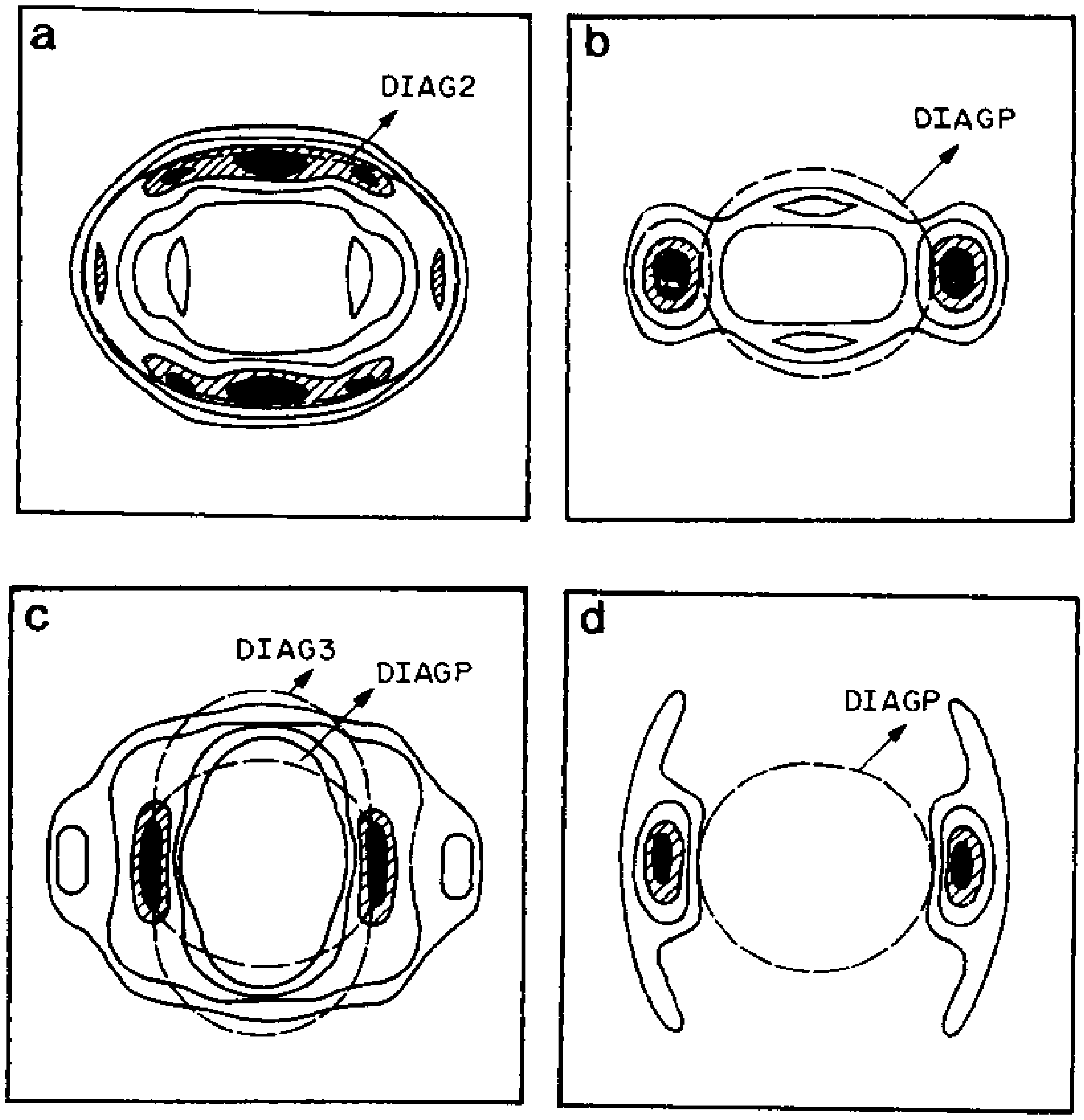

FIG. 4. (a)–(d) Same as Figs. 3(a–d) for $h_l(z)$. Classical periodic orbits are drawn.

areas can be interpreted as projections of the enhancements in the full Husimi distribution due to the self crossings due to the spin degree of freedom. Figure 4b exhibits an enhancement which we attribute to cusps and self crossings of orbits belonging to the family DIAGP. Figure 4c also seems to reflect signatures of DIAGP and DIAG3. Figure 4d shows an intensity enhancement which can be connected to orbits from the family DIAGP.

We have also analyzed the Husimi distributions for the case in Figs. 1a, as well as for $G' = 0$ and simply quote the results here. In the absence of chaos ($G' = 0$) both projections of the Husimi distribution show annular regions of concentration. This can be interpreted as being due to the presence of tori. When the energy is high enough we observe two regions of annular concentration in the spin projection, whereas the particle projection always has only one annular concentration. When a small amount of chaos is allowed ($G' = 0.2$) the Husimi distributions are essentially analogous to the ones we have presented in this work, in the sense that they have the same qualitative features. The transition from integrable to the chaotic situation (in Fig. 1b) seems to be smooth.

5. Concluding Remarks

In the present work we analyzed characteristic effects of finite Hilbert space systems in the nonintegrable situation. In the classical limit the projections of periodic orbits in the spin degree of freedom are restricted in phase space: they should be contained within a circle of radius $\sqrt{4J}$. When the energy is high enough such trajectories exhibit many self crossings and cusps (I), as a consequence of this limitation. The quantum analogue of this effect is a marked intensity enhancement of the wave functions with the corresponding energies. We believe this effect to be quite general and investigation of other spin systems are in progress.

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