THE GENEALOGY OF PERIODIC TRAJECTORIES

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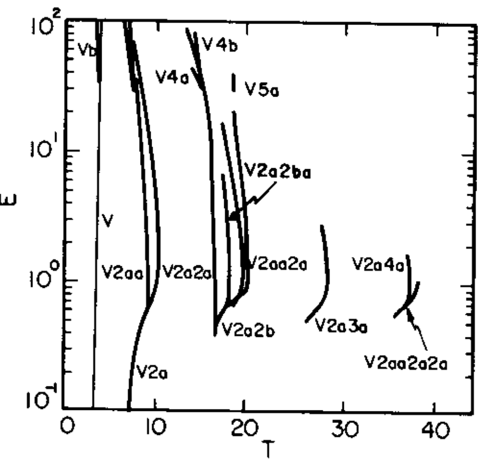
We have investigated numerically the periodic solutions of non-integrable classical Hamiltonian systems with two degrees of freedom. We obtained extensive numerical data and this was possible due to the development of new computational methods that are very fast and work very well independently of the periodic trajectory being stable or unstable. Our motivation for the present investigation was primarily to understand quantization as, in the study of many body nuclear systems, there are approximate methods that provide a classical description of collective modes. Therefore, quantization is required in order to describe bound-state spectrum or fluctuations. As is well known[1], the periodic trajectories form one-parameter families. Two convenient labelling parameters for a particular trajectory are its energy E or its period T. Most of our data are presented in the form E-T plots where each of the periodic families is represented by a line. The E-T plot provides a signature of the Hamiltonian H being studied, hence it is important to study the topology of the E-T plot.

The periodic trajectory is characterized by a matrix M called monodromy matrix[2]. For two dimensions this is a 4x4 matrix having two unit eigenvalues. The other two have unit product. The trajectory is stable if the eigenvalues have magnitude 1, therefore the trajectory is stable if the trace of M lies between 0 and 4.

The topology of the E-T plot is determined by its branchings. At an isochronous branching, M has four unit eigenvalues and $\mathrm{Tr}\,M = 4$. At a period doubling branching, two eigenvalues must be -1 and $\mathrm{Tr}M=0$. Period-Triplings occur for $\mathrm{Tr}\,M=1$, period-quadruplings for $\mathrm{Tr}\,M = 2$, etc.. Obviously, the E-T plot at large T can become very dense and complicated, but it will never be everywhere dense. The families are discrete and the most important ones are found at small T. We present here the results for the following Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}x^2 + \frac{3}{2}y^2 - x^2 y + \frac{1}{12}x^4 \quad .$$

It was chosen as a less symmetrical form of the Hénon-Heiles potential[3]. There are half dozen Hamiltonians under investigation but we do not expect the topological behaviour to depend essentially on the Hamiltonian used, thus we shall not mention results for the other Hamiltonians. Two of these families are obtained immediately: they correspond to harmonic oscillations of small amplitudes, around the equilibrium point, in the vertical and in the horizontal directions (normal modes). The vertical (V) family appears as a vertical line in the E-T plot (see fig. 1) because the potential V(0,y) is purely quadratic. The horizontal oscillation gives rise to the horizontal (H) family which has the period varying with amplitude. At low energy the H family is called boomerang (B) family (see fig. 1). The continuous line which represents a family on the E-T plot cannot begin or end except for two reasons: (1) it branches upon another family; (2) it becomes the family of small oscillations about an equilibrium point.



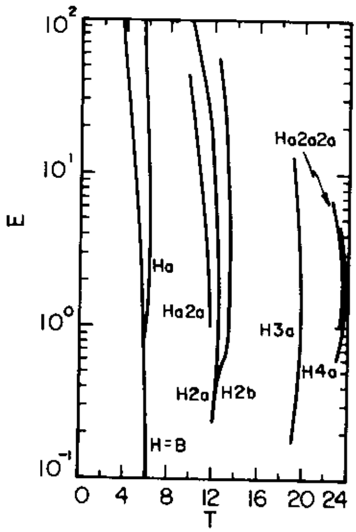


Figure 1 - V and H families and their branchings indicated by lower-case roman letters. The integer n before the letter indicates period n-pling (n=1 is omitted).

There are families that do not terminate anywhere: either they go on to infinity or they form closed curves (see fig. 2). The families S starting at the saddle points are always unstable with $TrM \rightarrow 4$ as $E \rightarrow \infty$. A family can exhibit more than one region of stability.

This happens for both the H and the V families.

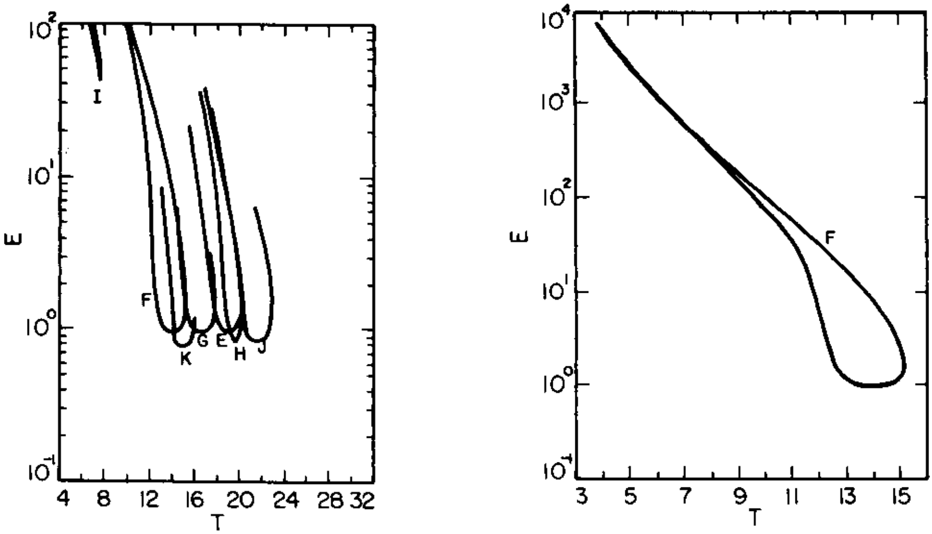


Figure 2 - Families that form closed curves.

Our main results are:

1) The vertical and the horizontal families are connected. For the potential above this connection happens via an isochronous branching (Vb = Hb) at E very high (23431).

2) At the points where TrM is tangent to zero or 4 there is a double branching, one stable and one unstable (see fig. 1).

3) For all the families that form closed curves TrM = 4 at the points where $\frac{dE}{dT} = 0$ and at these points there is no branching, the main trajectory switching simply from stable to unstable.

4) Period n-pling ($n \geqslant 3$) we believe gives rise to two distinct families, one stable and the other one unstable.

5) When two distinct families emerge at a branch point one of them is a libration and the other one is a rotation.

REFERENCES

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2 - L. Pontriaguine, Équations Différentielles Ordinaires, MIR, 1975.
3 - M. Hénon and C. Heiles, Astronomical Journal 69 (1964), 73.