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ISOCHRONOUS AND PERIOD DOUBLING BIFURCATIONS OF PERIODIC SOLUTIONS OF NON-INTEGRABLE HAMILTONIAN SYSTEMS WITH REFLEXION SYMMETRIES*

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We analytically derive the possible types of isochronous and period doubling bifurcations undergone by periodic solutions of two degrees of freedom, non-integrable, Hamiltonian systems possessing reflexion and time-reversal symmetries. We find that one of the isochronous bifurcations numerically found in refs. [3] *cannot* exist. In the case of period-doubling we predict the existence of a type of bifurcation not found in refs. [2] and [3] but confirmed by further numerical investigation.

1. Introduction

The periodic solutions of non-integrable two degrees of freedom Hamiltonians form one parameter families (the parameter is the energy E or the period τ). As the parameter is varied, one moves along the family and there exist values of this parameter at which new families of periodic solutions are generated. The points at which this happens are called bifurcation points (the parameter values at these points will be denoted by τ_b and E_b). The period of the bifurcated trajectories will be an integer multiple of the original trajectory i.e., $n\tau_b$. Isochronous and period-doubling bifurcations correspond to $n=1$ and 2, respectively. The periodic trajectory may or may not become hyperbolic when undergoing a bifurcation and the new families branching off the bifurcation point may be either elliptic or hyperbolic. The quantity conserved as the parameter varies across the bifurcation point is the so-called Poincaré index [1].

Extensive numerical analysis of period n-upling bifurcations of period solutions have been performed recently [2, 3] for non-integrable Hamiltonian systems with two degrees of freedom. Ref. [2] also presents an analytic study of part of the results obtained numerically. In this work this analytical investigation will be completed.

All systems investigated in refs. [2] and [3] were of the form

$$H(x, p_x, y, p_y) = \frac{p_x^2}{2} + \frac{p_y^2}{2} + V(x, y), \tag{1.1}$$

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therefore having time-reversal symmetry. Besides time-reversal symmetry, three of the systems also had the reflexion symmetry $x \to -x$. Due to time-reversal invariance, the periodic solutions of (1.1) are either librations (trajectories whose x–y projections have turning points so that they are followed in both directions) or rotations (trajectories whose x–y projections do not have turning points and are always followed in the same direction). A libration is its own time reverse while a rotation has for time reserve the trajectory followed in the opposite direction, which we consider here to be a different solution. If the Hamiltonian (1.1) also possesses some reflexion symmetry, then it may have for periodic solutions librations and rotations with the same reflexion symmetry (x-reflexion for systems in ref. [2]). Therefore, when $V(x, y)$ has one reflexion symmetry, the periodic solutions of (1.1) are of the following types [4]: *asymmetric rotation* (zero symmetry), *asymmetric libration* and *symmetric rotation* (one symmetry), *symmetric librations* (two symmetries). A study of the generic properties of mechanical systems possessing reflexion symmetries can be found in the paper by Devaney [5]. One should note that both symmetries considered here can be thought of as "reversing involutions" in the sense of Devaney [5] and all his results can be applied. Of particular interest here is the fact that, generically, symmetric periodic orbits lie on one-parameter families of closed orbits. Abundant numerical examples of such families can be found in refs. [2] and [3].

It was found numerically [2, 3] that periodic trajectories possessing symmetries exhibit different types of period n-upling bifurcations. The analytical study presented in ref. [2] accounted for *all* the period n-upling bifurcations of periodic trajectories possessing *zero* or *one* symmetry (i.e., bifurcations of asymmetric and symmetric rotations, and asymmetric librations). However, in the case of *symmetric librations*, only the period n-upling bifurcations with $n \geq 3$ were derived in ref. [2]. The isochronous ($n = 1$) and period-doubling ($n = 2$) bifurcations of *symmetric librations* which were not obtained analytically in ref. [2], will be derived in the present work.

In section 2 we derive the Poincaré map $\mathscr{P}$ appropriate for analyzing the isochronous and period doubling bifurcations of symmetric librations. It turns out that, for these trajectories, the Poincaré map can be written as the square of another map that we shall call $\mathscr{P}^{1/2}$. As will be shown in the next section, this is the fundamental property that one must take into account to properly analyze the bifurcations of such orbits. The basic techniques used here to construct these maps and find its fixed points follow the general ideas by Meyer [8]. The period-doubling bifurcation is presented in section 3. Our analysis predicts the existence of a new type of period-doubling bifurcation that was not contained in the numerical work of refs. [2] and [3]. And this prediction has been confirmed by further numerical investigation. In section 4 we present the isochronous bifurcations of symmetric librations. Our analysis *rules out* the existence of isochronous bifurcationsin which the original family remains elliptic. Such bifurcations were found in the numerical work of refs. [3], but we believe that these cases actually consist of two distinct bifurcations very close to each other. Indeed, a careful numerical investigation of one of them has shown it to be double, and therefore it has confirmed our analytical findings. Section 5 contains some final remarks and conclusions.

2. The "square-root" map $\mathscr{P}^{1/2}$

In the vicinity of a periodic solution, we may introduce periodic coordinates and transform the original time-independent Hamiltonian with two degrees of freedom into a time-dependent Hamiltonian with one degree of freedom [6]. The coordinate varying along the trajectory plays the role of time, its period being

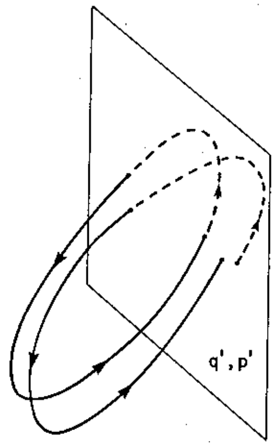


Fig. 1. Illustrating the Poincaré map of a section.

set equal to 2π. Using energy conservation,

$$H(x, p_x, y, p_y) = E, \tag{2.1}$$

the time-dependent reduced Hamiltonian is given by (choosing x to play the role of time)

$$h(q, p, t) \equiv p_x(y, p_y, E, -x), \tag{2.2}$$

where we have set $q \equiv y$, $p \equiv p_y$ and $t = -x$. Variation of the action restricted to the energy shell gives the following equations of motion in the vicinity of the periodic solution

$$\dot{q} = \frac{\partial h}{\partial p}, \quad \dot{p} = -\frac{\partial h}{\partial q}, \tag{2.3}$$

the dot meaning "time"-derivative.

And the Poincaré map of a section which is the map of a plane $x = \text{const.}$ (or $y = \text{const.}$) on itself, resulting from the consecutive intersections of this plane and the phase space trajectory with $p_x > 0$ ($p_y > 0$) (see fig. 1) is given by

$$(q(t), p(t)) \xrightarrow{\mathscr{P}} (q(t+2\pi), p(t+2\pi)), \tag{2.4}$$

with $q(t+2\pi)$ and $p(t+2\pi)$ solutions of (2.3) with initial conditions $q(t)$, $p(t)$. Thus, a periodic trajectory is a fixed point of $\mathscr{P}$ (see fig. 1). Therefore, bifurcations of periodic trajectories correspond to bifurcations of $\mathscr{P}$ and vice-versa so that the period n-upling bifurcations of periodic trajectories may be obtained by analyzing the fixed points of $\mathscr{P}^n$ at the bifurcation point [7]. Any fixed point of $\mathscr{P}$ is a trivial fixed point of $\mathscr{P}^n$ (it corresponds to going over the periodic trajectory n times). Period n-upling bifurcations will occur when $\mathscr{P}^n$ has at least one set of n fixed points besides the trivial one. The non-trivial fixed points of $\mathscr{P}^n$ may arise whenever the eigenvalues of the Jacobian of the linearized $\mathscr{P}$ are $e^{i2\pi(l/n)}$ and $e^{-i2\pi(l/n)}$, with l an integer non-commensurate with n [7]. This means that isochronous bifurcation occurs when the Jacobian of the linearized $\mathscr{P}$ has eigenvalues 1 and period-doubling bifurcation occurs when it has eigenvalues -1.

The map $\mathscr{P}$ is area preserving and has the same symmetries of the reduced Hamiltonian (2.2). Therefore, it has the symmetry $p \to -p$ (the Poincaré map for the section $y = \text{const.}$ has two symmetries: $x \to -x$ and $p_x \to -p_x$).

Now, if the periodic trajectory is a symmetric liberation, the fixed points of the Poincaré map of section $\mathscr{P}$ are also fixed points of the area preserving "square root" map $\mathscr{P}^{1/2}$ of the plane (q, p) on itself, defined as

$$(q(t), p(t)) \xrightarrow{\mathscr{P}^{1/2}} (q(t+\pi), p(t+\pi)), \tag{2.5}$$

where $q(t+\pi)$ and $p(t+\pi)$ are solutions of (2.3) with initial conditions $q(t), p(t)$. This "square root" map will also have the symmetry $p \to -p(p_y \to -p_y)$.

Remark. The map $\mathscr{P}^{1/2}$ is not a map of a Poincaré section as phase space points having $p_x > 0$ are mapped onto phase space points having $p_x < 0$.

And the Poincaré map $\mathscr{P}$ for a symmetric libration can be written as

$$\mathscr{P} = \mathscr{P}^{1/2} \cdot \mathscr{P}^{1/2}. \tag{2.6}$$

The above composability is the property characterizing the Poincaré map for a symmetric libration.

The eigenvalues of the linearized map $\mathscr{P}^{1/2}$ are the square root of the eigenvalues of the linearized $\mathscr{P}$; therefore, at the bifurcation point, the linearized map $\mathscr{P}^{1/2}$ giving rise to isochronous bifurcation must have eigenvalues 1 or -1. In the case of period doubling, it must have eigenvalues $e^{\pm i(\pi/2)}$. These $\mathscr{P}^{1/2}$ maps will be constructed in sections 3 and 4.

Before going on to the next section, a few words about stability of the trajectories are in order. The periodic trajectories will have the same stability as the fixed points. Therefore, elliptic trajectories correspond to elliptic fixed points and hyperbolic trajectories correspond to hyperbolic fixed points. At elliptic (hyperbolic) fixed points, the eigenvalues of the Jacobian of the linearized map are $e^{\pm i\alpha}(e^{\pm\alpha})$, α real. Note that elliptic or hyperbolic do not mean stability or instability, respectively: it is well-known [8] that in Hamiltonian systems a trajectory can be unstable even though its eigenvalues have unity modules. In fact, every time the eigenvalues are of type $e^{\pm 2\pi i r/s}$ with r and s integers, the periodic trajectory corresponding to s windings around the original one will have eigenvalues $e^{\pm 2\pi i r} = 1$, which is exactly the boundary between stability and instability. In this case one has to go to non-linear terms in the perturbation expansion in order to determine the stability character, instead of stopping in the linearized map. A classical example, discussed in detail by Arnold [9], is the period tripling point, where the eigenvalues are $e^{\pm 2\pi i/3}$.

3. Period-doubling bifurcations

Following the prescription of the last section, we shall express the Poincaré map $\mathscr{P}$ in terms of map $\mathscr{P}^{1/2}$. For the period-doubling case, as already mentioned, the eigenvalues of the Jacobian of $\mathscr{P}^{1/2}$ must be $e^{\pm i(\pi/2)}$ at the trivial fixed point at $E = E_b$. Setting $E = E_b + \epsilon$ (so that the bifurcation point corresponds to $\epsilon = 0$) and taking the origin as the trivial fixed point, the general polynomial expansion for the map $\mathscr{P}^{1/2}$,

$$\begin{pmatrix} q_{1/2} \\ p_{1/2} \end{pmatrix} = \mathscr{P}^{1/2} \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}, \tag{3.1}$$

is [7]

$$\begin{aligned} q_{1/2} &= p_0 + Aq_0 + \epsilon(a_0 + a_1 q_0 + a_2 p_0) + a_{11} q_0^2 + a_{12} q_0 p_0 \\ &\quad + a_{22} p_0^2 + a_{30} q_0^3 + a_{31} q_0^2 p_0 + a_{32} q_0 p_0^2 + a_{33} p_0^3 + \cdots, \\ p_{1/2} &= -q_0 + Bq_0 + \epsilon(b_0 + b_1 q_0 + b_2 p_0) + b_{11} q_0^2 + b_{12} q_0 p_0 \\ &\quad + b_{22} p_0^2 + b_{30} q_0^3 + b_{31} q_0^2 p_0 + b_{32} q_0 p_0^2 + b_{33} p_0^3 + \cdots. \end{aligned} \tag{3.2}$$

Area preservation plus the t-reflection symmetry $p \to -p$ imposed on (3.2) imply that $A = B = 0$, $b_0 = a_0$, $b_2 = b_1 = a_1 = a_2$, $a_{22} = a_{12} = b_{22} = b_{12} = b_{11} = a_{11}$, $b_{30} = a_{33}$, $b_{31} = a_{32}$, $a_{31} = b_{32} = 3a_{33}$ and $b_{33} = a_{30}$ and, up to third order, the map $\mathscr{P}^{1/2}$ is given by

$$\begin{aligned} q_{1/2} &= p_0 + \epsilon[a_0 + a_2(q_0 + p_0)] + a_{11}(q_0 + p_0)^2 + a_{30} q_0^3 + 3a_{33} q_0^2 p_0 + a_{32} q_0 p_0^2 + a_{33} p_0^3, \\ p_{1/2} &= -q_0 + \epsilon[a_0 + a_2(q_0 + p_0)] + a_{11}(q_0 + p_0)^2 + a_{33} q_0^3 + a_{32} q_0^2 p_0 + 3a_{33} q_0 p_0^2 + a_{30} p_0^3. \end{aligned} \tag{3.3}$$

In order to analyze the fixed points of $\mathscr{P}^{1/2}$ we define the following variables,

$$q = \sqrt{\epsilon}\, r, \quad p = \sqrt{\epsilon}\, t, \tag{3.4}$$

in terms of which the map (3.3) becomes a power series expansion in the ϵ parameter [10]:

$$\begin{aligned} r_{1/2} &= t_0 + \sqrt{\epsilon}\left[a_0 + a_{11}(r_0 + t_0)^2\right] + \epsilon\left[a_2(r_0 + t_0) + a_{30} r_0^3 + 3a_{33} r_0^2 t_0 + a_{32} r_0 t_0^2 + a_{33} t_0^3\right] + \mathscr{O}(\epsilon^2), \\ t_{1/2} &= -r_0 + \sqrt{\epsilon}\left[a_0 + a_{11}(r_0 + t_0)^2\right] \\ &\quad + \epsilon\left[a_2(r_0 + t_0) + a_{33} r_0^3 + a_{33} r_0^2 + a_{32} r_0^2 t_0 + 3a_{33} r_0 t_0^2 + a_{30} t_0^3\right] + \mathscr{O}(\epsilon^2). \end{aligned} \tag{3.5}$$

And substitution of (3.5) in (2.6) yields the Poincaré map for the period-doubling case:

$$\begin{aligned} r_1 &= -r_0 + 2\sqrt{\epsilon}\left[a_0 + a_{11}(r_0^2 + t_0^2)\right] \\ &\quad + 2\epsilon\left[\gamma t_0 - 2r_0 a_0 a_{11} + a_{32} r_0^2 t_0 + a_{30} t_0^3 + 2a_{11}^2 (t_0 - r_0)(r_0 + t_0)^2\right] + \mathscr{O}(\epsilon^2), \\ t_1 &= -t_0 - 4a_{11}\sqrt{\epsilon}\, r_0 t_0 - 2\epsilon\left[\gamma r_0 - 2t_0 a_0 a_{11} + a_{32} r_0 t_0^2 + a_{30} r_0^3 - 2a_{11}^2 (t_0 - r_0)(r_0 + t_0)^2\right] + \mathscr{O}(\epsilon^2), \end{aligned} \tag{3.6}$$

where

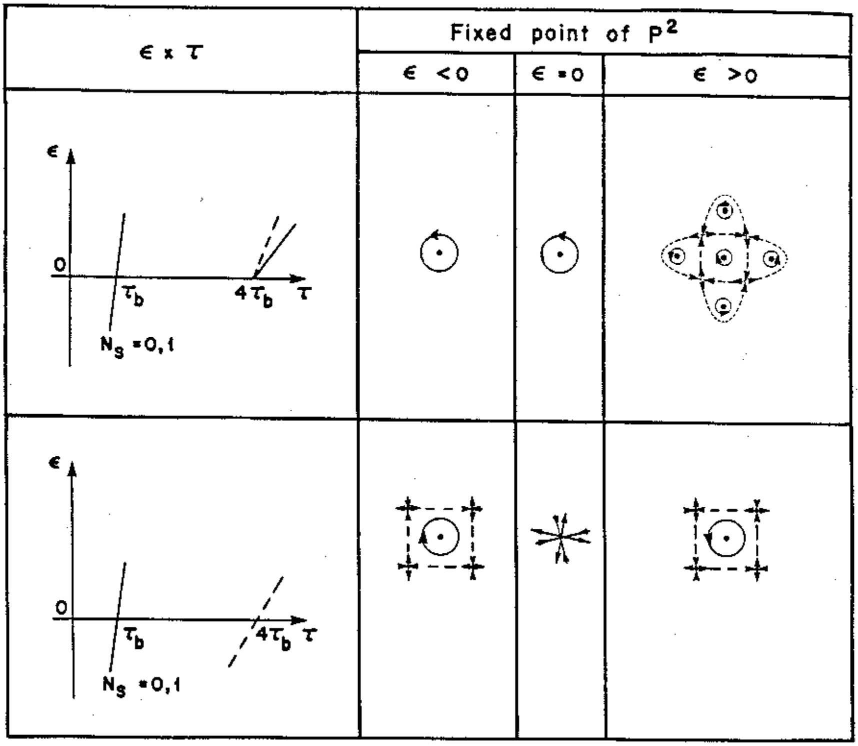
$$\gamma = a_2 + 2a_0 a_{11}. \tag{3.7}$$

Using the implicit function theorem [11] we obtain the fixed points of (3.5) and (3.6). The only fixed point that $\mathscr{P}$ and $\mathscr{P}^{1/2}$ have in common is

$$r = a_0\sqrt{\epsilon}, \quad t = 0. \tag{3.8}$$

Remark. This fixed point would be immediately obtained if we had used the transformation $p = \sqrt{\epsilon}\, t$, $q = \epsilon r$ instead of (3.4). However, if we use this transformation we will not be able to obtain the existing

Table I
$\epsilon \times \tau$ plot and fixed points (with map invariant curves) at period doubling bifurcation points. A full (dashed) line is used here to indicate a elliptic (hyperbolic) family. Thick lines (full or dashed) indicate that there are two degenerate families branching off the bifurcation point.



non-trivial fixed points of $\mathscr{P}^2$ as they are of order $\sqrt{\epsilon}$ and the resulting expansion will in this case start with order ϵ.

The eigenvalues of the Jacobian of the linearized map (3.6) at the fixed point (3.8) are

$$\lambda = -1 \pm 2\mathrm{i}\epsilon\gamma, \tag{3.9}$$

with γ given by (3.7). So, the fixed point of $\mathscr{P}$ is an elliptic fixed point in the vicinity of $\epsilon = 0$. Therefore, the corresponding trajectory remains always elliptic when ϵ goes across zero (the bifurcation point).

The $\mathscr{P}^2$ map is obtained by iterating (3.6). The result is

$$\begin{aligned} r_2 &= r_0 - 4\epsilon t_0\left(\gamma + \alpha r_0^2 + \beta t_0^2\right) + \mathscr{O}(\epsilon^2), \\ t_2 &= t_0 + 4\epsilon r_0\left(\gamma + \beta r_0^2 + \alpha t_0^2\right) + \mathscr{O}(\epsilon^2), \end{aligned} \tag{3.10}$$

where

$$\alpha = a_{32} - 2a_{11}^2, \quad \beta = a_{32} + 2a_{11}^2. \tag{3.11}$$

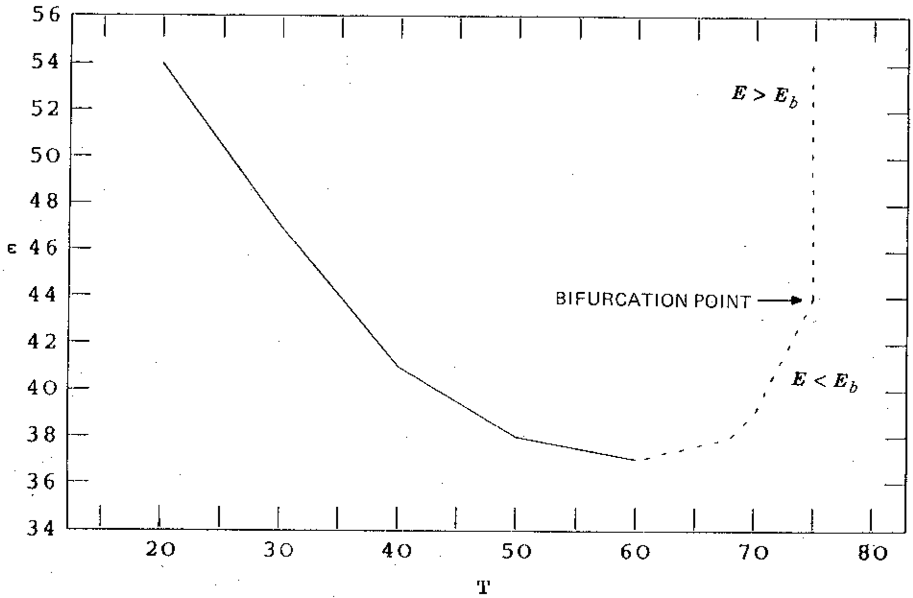


Fig. 2. $E \times \tau$ plot ($\tau = 38.36 + 10^{-5}T$, $E = 0.9576769 + 10^{-9}\epsilon$) at a period doubling bifurcation of the MARTA potential with the two pairs of bifurcated trajectories being hyperbolic, one pair existing only for $E > E_b$, the other only for $E < E_b$. See caption of table I for notation.

The above map has the following nine fixed points:

$$\mathrm{I)}\quad \begin{cases} r = 0, \\ t = 0; \end{cases} \tag{3.12}$$

$$\mathrm{II)}\quad \begin{cases} r = 0, \\ t = \pm\sqrt{\dfrac{-\gamma}{\beta}}\,; \end{cases} \quad \text{and} \quad \begin{cases} r = \pm\sqrt{\dfrac{-\gamma}{\beta}}\,, \\ t = 0; \end{cases} \tag{3.13}$$

$$\mathrm{III)}\quad \begin{cases} r = \pm\sqrt{\dfrac{-\gamma}{\alpha+\beta}}\,, \\ t = \pm\sqrt{\dfrac{-\gamma}{\alpha+\beta}} \end{cases}, \tag{3.14}$$

with γ, α, β given by (3.7) and (3.11), respectively.

The eigenvalues of the Jacobian of the linearized $\mathscr{P}^2$ map (3.10), calculated at the fixed points (3.11), (3.12) and (3.14) are, respectively,

$$\lambda_{\mathrm{I}} = 1 \pm 4\mathrm{i}\epsilon\gamma, \tag{3.12a}$$

$$\lambda_{\mathrm{II}} = 1 \pm 4\sqrt{2}\,\gamma a_{11}\epsilon/\sqrt{\beta}\,, \tag{3.13a}$$

$$\lambda_{\mathrm{III}} = 1 \pm \frac{16\sqrt{2}\,\gamma a_{11}\epsilon}{\alpha+\beta}\sqrt{2a_{11}^2 - \beta}\,. \tag{3.14a}$$

From (3.12a) one sees that the fixed point (3.12) is always elliptic. This is also a fixed point of $\mathscr{P}$ at $\epsilon=0$ (the bifurcation point). Therefore, it corresponds to the periodic trajectory of period τ_b at $\epsilon=0$. The remaining fixed points are *only* fixed points of $\mathscr{P}^2$, therefore, each pair corresponds to a periodic trajectory of period $2\tau_b$ at $\epsilon=0$. So, assuming $\gamma<0$ (the case $\gamma>0$ gives the same results) there are three possibilities:

(i) If $\beta<0$ the pair of trajectories corresponding to the two sets of fixed points given in (3.13) are elliptic while the pair corresponding to the two sets of fixed points (3.14) are hyperbolic (see (3.13a) and (3.14a)). Both pairs exist only for $\epsilon<0$. In the first row of table I we show the $\epsilon \times \tau$ plot for this case in the vicinity of $\epsilon=0$, together with the fixed points and invariant curves of $\mathscr{P}^2$.

(ii) If $0<\beta<2a_{11}^2$, the pair of trajectories corresponding to the sets of fixed points (3.13) is hyperbolic and so is the pair of trajectories corresponding to the set of fixed points (3.14) (see (3.13a) and (3.14a)). One of the pairs existing for $\epsilon<0$ only and the other pair existing for $\epsilon>0$ only. The $\epsilon \times \tau$ plot for this case, together with the fixed points and invariant curves of $\mathscr{P}^2$ are shown in the second row of table I.

(iii) If $\beta>2a_{11}^2$, the pair of trajectories corresponding to fixed points (3.13) and (3.14), respectively, will exist only for $\epsilon>0$, the pair corresponding to (3.13) being hyperbolic and the pair corresponding to (3.14) being elliptic (see (3.13a) and (3.14a)). This is the same as in case (i) with $\epsilon \to -\epsilon$.

Summarizing our results: when a *symmetric liberation* undergoes a period doubling bifurcation it *always* remains elliptic and two pairs of periodic trajectories with twice its period are generated that are either both hyperbolic, one pair existing for $\epsilon>0$ the other for $\epsilon<0$, or one pair elliptic and another hyperbolic, both existing only for $\epsilon>0$ (or $\epsilon<0$).

Case (i) (or (iii)) were obtained numerically in refs. [2] and [3]. It is observed at the bifurcations occurring at points labeled Z^2 in those references. Case (ii) was not obtained in those numerical investigations but after it was predicted by our analysis we did a numerical check and confirmed its existence.

In fig. 2 we present the $E \times \tau$ plot of one of these bifurcations for the MARTA potential [2],

$$H(x, y)=\frac{p_x^2}{2}+\frac{p_y^2}{2}+\frac{x^2}{2}+\frac{3y^2}{2}-x^2y+\frac{x^4}{12}.$$

This was also checked for one of the potentials of ref. [3] (the NELSON potential) [12].

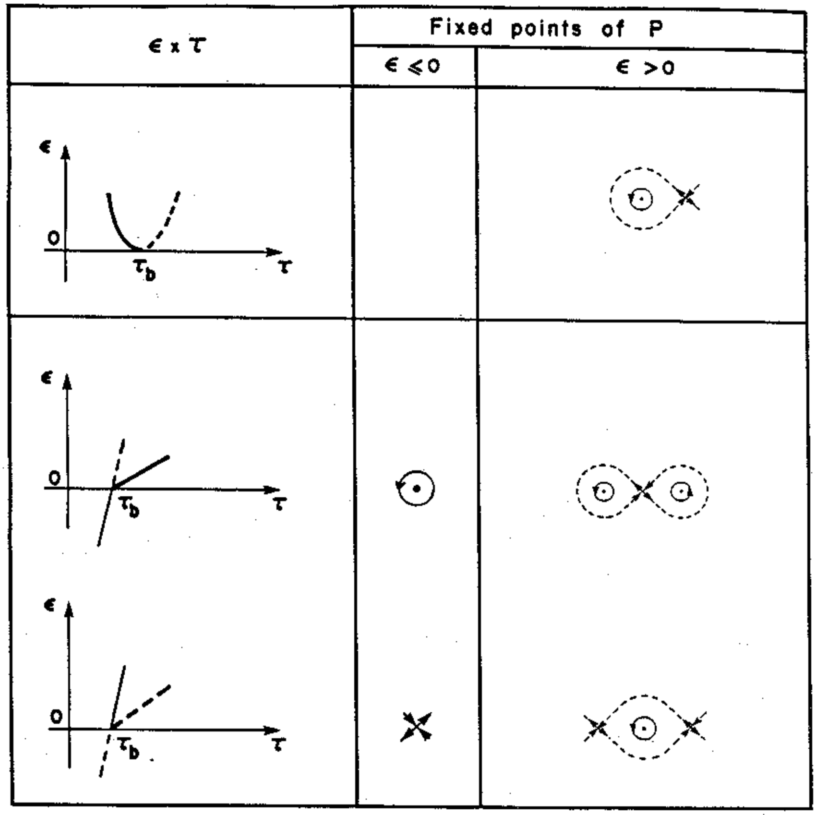
4. Isochronous bifurcations

In the case of an isochronous bifurcation, there are three possibilities for the "square root" map $\mathscr{P}^{1/2}$:

$$\begin{aligned} q_{1/2}&=q_0+\epsilon\left[p_0+(1+b_0a_{12})q_0/2\right]+a_{12}\left(q_0^2/2+q_0p_0\right),\\ p_{1/2}&=p_0+q_0+\epsilon\left[b_0+b_1q_0+(1-b_0a_{12})p_0/2\right]+b_{11}q_0^2-a_{12}p_0^2/2, \end{aligned} \tag{4.1}$$

$$\begin{aligned} q_{1/2}&=q_0+p_0+\epsilon(q_0+a_2p_0+b_0/2)+4a_{22}q_0p_0+a_{11}q_0^2+a_{22}p_0^2,\\ p_{1/2}&=p_0+\epsilon\left[b_0+2(a_1-a_0a_{12}+a_0a_{11})q_0+(a_1-2a_0a_{12}+2a_0a_{11})p_0\right]\\ &\quad+2a_{11}q_0(q_0+p_0)+(a_{11}-2a_{22})p_0^2, \end{aligned} \tag{4.2}$$

Table II
$\epsilon \times \tau$ plot and fixed points (with map invariant curves) at isochronous bifurcation points. See caption of table I for notation.



and

$$
\begin{aligned}
q_{1/2} &= -q_0 + \epsilon\left(a_0 + a_1 q_0 + p_0\right) + a_{11} q_0^2 + a_{12}\, p_0\left(q_0 - p_0\right), \\
p_{1/2} &= -p_0 + q_0 + \epsilon\left[-a_0/2 + b_1 q_0 - \left(1 + a_1\right) p_0\right] + \left(2a_{11} + a_{12}\right)\left(q_0^2/4 - q_0 p_0\right) + a_{12}\, p_0^2/2,
\end{aligned}
\tag{4.3}
$$

where $a_0(2a_{11} + a_{12}/2) = -(1 + 2a_1)$.

The maps (4.1) and (4.2) are the maps we obtained in ref. [2] for the isochronous bifurcation of *symmetric rotations or asymmetric liberation* (trajectories having *one* symmetry). And map (4.3) is the map obtained by us in ref. [2] for describing the period doubling bifurcations.

The map expansion in powers of the parameter ϵ is obtained for (4.1) and (4.3) by using the variables

$$p = \sqrt{\epsilon}\, t, \quad q = \epsilon r. \tag{4.4}$$

For map (4.2) we use the variables

$$p = \sqrt{\epsilon}\, t, \quad q = \sqrt{\epsilon}\, t. \tag{4.5}$$

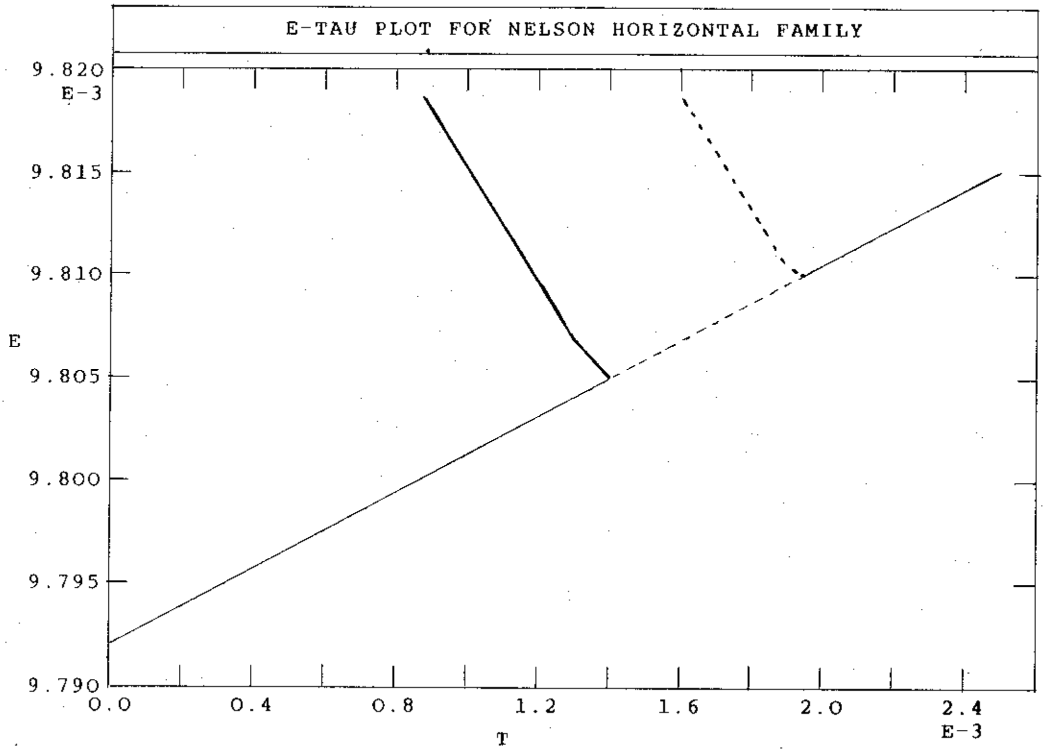


Fig. 3. $E \times \tau$ plot ($\tau = T + 21.0000$) showing two consecutive isochronous bifurcations occurring in the NELSON potential [3]. Notice that if the resolution is not good enough, the bifurcating trajectory will seem to be always elliptic. See caption of table I for notation.

After the above change of variables we obtain the three possible maps $\mathscr{P}$ by iterating the corresponding map $\mathscr{P}^{1/2}$. The result is

$$\begin{aligned} r_1 &= r_0 + 2\sqrt{\epsilon}\, t_0\left(1 + a_{12} r_0 + a_{33} t_0^2\right) + \mathcal{O}(\epsilon), \\ t_1 &= t_0 + 2\sqrt{\epsilon}\left(r_0 + b_0 - a_{12} t_0^2/2\right) + \mathcal{O}(\epsilon), \end{aligned} \tag{4.1a}$$

$$\begin{aligned} r_1 &= r_0 + 2t_0 + 4\sqrt{\epsilon}\left[b_0/2 + a_{11} r_0^2 + \left(a_{22} + a_{11}/2\right) t_0^2 + \left(2a_{22} + a_{11}\right) r_0 t_0\right] + \mathcal{O}(\epsilon), \\ t_1 &= t_0 + 2\sqrt{\epsilon}\left[b_0 + 2a_{11} r_0^2 + 4a_{11} r_0 t_0 + t_0^2\left(3a_{11} - 2a_{22}\right)\right] + \mathcal{O}(\epsilon), \end{aligned} \tag{4.2a}$$

and

$$\begin{aligned} r_1 &= r_0 - 2\sqrt{\epsilon}\, t_0\left[\left(1 + a_0 a_{12}\right) + \left(a_{33} - a_{12}^2\right) t_0^2 - a_{12} r_0\right] + \mathcal{O}(\epsilon), \\ t_1 &= t_0 - 2\sqrt{\epsilon}\left(r_0 - a_0/2 + a_{12} t_0^2/2\right) + \mathcal{O}(\epsilon). \end{aligned} \tag{4.3a}$$

Examining the above expressions we see that (4.1a) and (4.2a) have the same qualitative features of the corresponding $\mathscr{P}^{1/2}$ map, (4.1) and (4.2). Also, the map (4.3a) is essentially the same as (4.1a). Since maps (4.1) and (4.2) have already been analyzed in ref. [2] (isochronous bifurcations of asymmetric librations or symmetric rotations) we shall only list the results.

Maps (4.1a) or (4.3a) have three fixed points, one corresponding to the main trajectory and the others corresponding to two bifurcated trajectories. At the bifurcation point the main trajectory changes from elliptic to hyperbolic (or vice-versa) and the two bifurcated trajectories have the same character (elliptic or hyperbolic) (see table II).

Map (4.2a) corresponds to the generic [10] case; there is no bifurcation and the trajectory simply switches from elliptic to hyperbolic or vice-versa. And it only exists either for $\epsilon > 0$ or for $\epsilon < 0$.

These are the *only* possible bifurcations of *symmetric librations*. In the first two refs. of [3] a bifurcation point labeled 4^2 was found numerically, at which a *symmetric libration* would remain elliptic when undergoing an isochronous bifurcation, with two pairs of trajectories being generated, one pair elliptic, the other hyperbolic. According to our results, this kind of bifurcation *cannot occur* for symmetric librations. Therefore, we decided to make a careful numerical investigation of the point 4^2 found in one of the potentials studied in refs. [3] (NELSON potential). And indeed, our numerical investigation has confirmed our present analysis. What happens is that two isochronous bifurcations occur in succession. Fig. 3 shows the $E \times \tau$ plot of these numerical results.

As we can see, the main trajectory becomes hyperbolic for a very small energy (period) interval. We believe that this interval of instability is so narrow because we are at so low an energy that the system is almost integrable and therefore should behave very much like an integrable one, which, as is well-known, does not have unstable regions in the bounded domain.

5. Conclusions

Reflexion symmetries lead to important qualitative changes in the bifurcations of periodic solutions of non-integrable Hamiltonian systems with two degrees of freedom. The relevant parameters for the classification of period n-upling bifurcation of a given period orbit are the multiplicity n and the number of symmetries N_s of the orbit. In our study we restricted ourselves to a maximum of two reflexion symmetries, one being time-reversal and the other the reflexion with respect to one spatial axis (x-reflexion has been considered). We found that if $N_s = 1$, (*asymmetric librations or symmetric rotations*) there exists a new type of isochronous ($n = 1$) bifurcation [2] not observed in the case of orbits with $N_s = 0$ (generic case [2, 10]). And no new features are observed [2] if $n \geq 2$. When an extra symmetry is present, i.e. $N_s = 2$ (the trajectories are *symmetric librations*), new types of period n-upling bifurcation will be observed only for $n > 1$. We found that the presence of an extra symmetry in the orbit does not lead to new types of isochronous bifurcations. Therefore, the isochronous bifurcation obtained numerically at points denoted by 4^2 in refs. [3] simply *cannot* exist, a fact confirmed by careful numerical investigation of one of these 4^2 points. Thus, when undergoing isochronous bifurcation, the periodic trajectory *always* changes from elliptic to hyperbolic or vice-versa.

In the case of period-doubling bifurcation we should note that the $\mathscr{P}^{1/2}$ map is like the map $\mathscr{P}$ used in ref. [2] to describe the period quadrupling of trajectories having one symmetry so that *period-doubling of symmetric librations* is like *period-quadrupling of symmetric rotations or asymmetric librations*.

Finally, we would like to make a remark about the use of numerical data: it should always be kept in mind that some properties cannot be observed unless the numerical resolution is high enough.

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