Z. Phys. A – Atomic Nuclei 325, 491–492 (1986)

Zeitschrift für Physik A
Atomic Nuclei

Short Note

Periodic Trajectories for Non-Integrable Two-Dimensional Hamiltonians

M. Baranger[1], **K.T.R. Davies**[1,2], **M.A.M. de Aguiar**[3], **C.P. Malta**[3], **and E.J.V. de Passos**[3]

[1] Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
[2] Physics Division, Oak Ridge National Laboratory, Oak Ridge, TN 37831, USA
[3] Instituto de Física Universidade de São Paulo, C.P. 20516, São Paulo, Brazil

Received November 4, 1986

Abstract: The potential used was motivated by collective motion in nuclear physics. A large number of numerical results have been organized into E-τ plots, whose branchings and topology have been studied. Several general rules are inferred.

We have obtained extensive numerical data concerning the periodic classical trajectories of a particle moving in a non-integrable two-dimensional potential. The details of the computational method will appear elsewhere;[1] a new aspect is that it works as well for unstable as for stable periodic trajectories. It has been recognized[2] that unstable periodic trajectories can play an important role in the quantization problem.

A periodic trajectory has an energy E and a period τ. It is well known[3] that periodic trajectories occur in one-parameter families, which define a continuous curve in the E-τ plane. Each Hamiltonian is characterized by the geometry and the topology of its E-τ plot; see Fig. 1. Another property of each periodic trajectory is its 4×4 monodromy matrix[4] M. Two eigenvalues of M are always unity.[4] The other two have unit product. If they are complex conjugates, the trajectory is stable and $0 < \mathrm{TrM} < 4$; if they are real, the trajectory is unstable and $\mathrm{TrM} < 0$ or > 4.

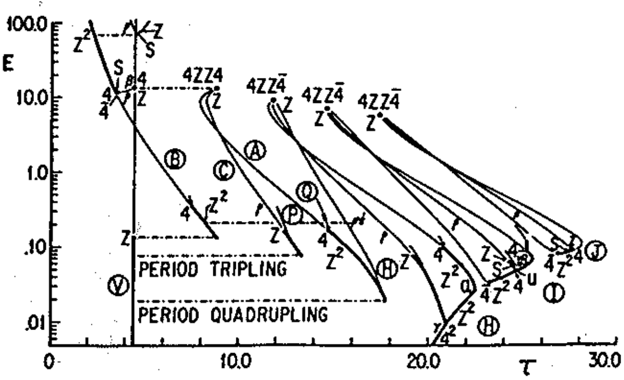

Fig. 1. E-τ plot of the main symmetric families, showing also the beginnings of asymmetric branchings. The circled capital letters are the family names.

The topology of the E-τ plot is determined by its branchings. At an isochronous branching there is a confluence of two distinct families and M has four unit eigenvalues, TrM = 4. At a period-doubling branching, two eigenvalues must be −1; hence, TrM = 0. There are also period-triplings (TrM = 1), period-quadruplings (TrM = 2), and so on. (See Fig. 1).

Most of our work was done with the Hamiltonian

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \left(y - \frac{x^2}{2}\right)^2 + \mu\,\frac{x^2}{2} \qquad (\mu = 0.1)$$

The potential consists of a deep valley in the shape of a parabola, surrounded by high mountains. The deep valley could represent a collective degree of freedom which, however, remains coupled to other types of excitation.

Time-reversal invariance dictates that there be two kinds of periodic trajectories, traditionally known as librations and rotations. On Fig. 1 we label the rotations by ρ; all others are librations. We use heavy lines or the symbol s for stable trajectories, thin lines or u for unstable ones. In general, stable trajectories lie in regular regions of phase space and unstable ones in chaotic regions.[5] The limiting values of TrM, 4 and Z ("zero"), are also marked on the plot, with 4^2 and Z^2 for double solutions. On Fig. 1 the period multiplying branchings are shown connected to their parent family by a dot-dashed horizontal line.

At small amplitudes there are two harmonic oscillations, one horizontal, one vertical, with two non-congruent periods. Note, however, that the potential on the y-axis is purely quadratic and therefore the vertical oscillations (V family) retain their linear harmonic behavior and their constant period for all amplitudes; they appear in Fig. 1 as a vertical line. Not so the horizontal oscillations (H family); Fig. 2 shows what happens to them as their amplitude increases. A naive expectation might have been that this family, as it grows, occupies the bottom of the potential valley and can be identified with the collective oscillations of nuclear physics. Figure 2 shows that this is true for a while but, suddenly, the trajectories start leaving the valley to climb the walls. This corresponds to a sharp left turn in the E-τ plot (symbol α of Fig.

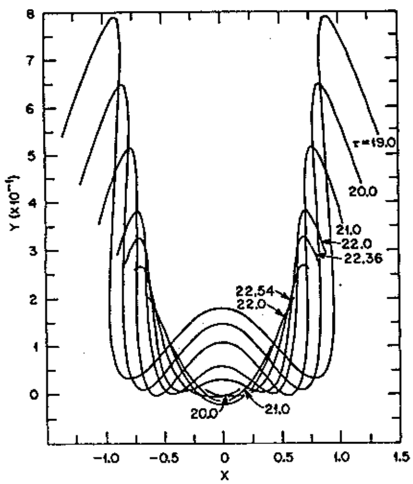

Fig. 2. Actual trajectories in the xy plane, for some low-energy members of the H family.

1). Another family, the I family, comes down from the mountains then, occupies the valley for a while, and returns to the mountains. Later, the J family comes down, stays for a while, and goes back up, etc. The real picture is thus vastly more complicated than a single, continuous, collective family. Note that in Fig. 1 there is a sizable gap between the H and I families, and another between I and J, so that there are values of τ for which no valley trajectory exists.

An interesting topological question concerns the connectedness of the E-τ plot. Fig. 1 shows that H and V are connected in several ways, B and A being part of the same cluster. As far as we know, I and J, together with their associated rotations and asymmetric librations, are not connected to this cluster; they constitute two islands. This could still be wrong if a connection existed through the period-multiplied trajectories, which is a problem we have not investigated. If they are islands, however, then it is not possible to generate the entire E-τ plot by the systematic procedure which consists of starting at a known point and following all the branches by continuity.

There may be more than one stable region within a given family. An obvious example is the B family (Fig. 1) which starts stable at low energy, has some important branchings, becomes unstable until E = 11.8, then enjoys another extended region of stability, all the way to infinity. Similarly, the V family has very small recurring regions of stability at higher and higher energies. This fact means that the standard scenario[6] in which the stability gets passed on by period-doubling to successively more complicated trajectories, though very attractive in its universality, can miss a lot of the information.

The standard scenario suggests that there might be something like "conservation of stability" at a branching. Let us call a "channel" any one of the E-τ lines issuing from a branching, including as separate channels both halves of the family which continues through the branching, and including period-multiplying. The simplest branchings have three channels and some have more. Then, conservation of stability would mean this: the number of stable channels emanating from a branching must be even, and is usually equal to two. We found this statement to be definitely false for some isochronous branchings. On Fig. 1 there are two cases having only one stable channel and one case having three. For simple period-doubling branchings (i.e., a single Z) the statement is probably always true. For Z^2 branchings, the statement is probably always false. For period-multiplyings higher than two, the statement is also false. The general rule for this case seems to be that there are two period-multiplied branches, one stable and one unstable, in addition to the two stable channels of the original family. In short, we find that the standard period-doubling scenario does seem to occur with conservation of stability, but it omits consideration of other effects in which stability is not conserved.

Moreover, we found many cases where TrM = 4 and there is no branching at all. It happens in all places, without exception, where the E-τ curve has a horizontal tangent. At these points the trajectories switch from stability to instability, but there is no branching. We designate these points on Fig. 1 by $\overline{4}$. There, the M matrix has a single eigenvector, the vector corresponding to an infinitesimal time-displacement along the trajectory. At a branching 4, it has two distinct eigenvectors, only one of which is the time-displacement vector.

Many of our results are in agreement with the analysis of Meyer.[2] However, he obtains a different branching behavior for period tripling and period quadrupling.

1. M. Baranger and K.T.R. Davies, in preparation.
2. M. C. Gutzwiller, J. Math. Phys. **11**, 1791 (1970); J. Math. Phys. **12**, 343 (1971); Phys. Rev. Lett. **45**, 150 (1980); R. Balian and C. Bloch, Ann. Phys. (N.Y.) **69**, 76 (1972); M. V. Berry, in Chaotic Behavior of Deterministic Systems - Les Houches 1981, eds. G. Iooss, R.H.G. Helleman, and R. Stora (North-Holland, Amsterdam, 1983); E. H. Heller, Phys. Rev. Lett. **53**, 1515 (1984).
3. V. I. Arnold, Mathematical Methods of Classical Mechanics (Springer-Verlag, New York, 1978), p. 389.
4. V. A. Yakubovich and V. M. Starzhinskii, Linear Differential Equations with Periodic Coefficients (Keter Publishing House, Jerusalem, 1975), especially Chaps. II and III.
5. V. I. Arnold and A. Avez, Ergodic Problems of Classical Mechanics (Benjamin, New York, 1968), Appendix 34.
6. P. Collet, J. P. Eckmann, and H. Koch, Physica 3D, 457 (1981); J. M. Greene, R. S. MacKay, F. Vivaldi, and M. J. Feigenbaum, Physica 3D, 468 (1981).
7. K. R. Meyer, Trans. of the Amer. Math. Soc. **149**, 95 (1970).