Dynamical Response of Networks under External Perturbations: Exact Results

David D. Chinellato[1], Irving R. Epstein[2,3], Dan Braha[2,4],

Yaneer Bar-Yam[2] and Marcus A.M. de Aguiar[1,2]

[1]*Instituto de Física 'Gleb Wataghin',*

Universidade Estadual de Campinas, Unicamp

13083-970, Campinas, SP, Brasil

[2]*New England Complex Systems Institute,*

Cambridge, Massachusetts 02138

[3]*Department of Chemistry, MS015,*

Brandeis University, Waltham,

Massachusetts 02454, USA

[4]*University of Massachusetts,*

Dartmouth, Massachusetts 02747

Abstract

We give exact statistical distributions for the dynamic response of influence networks subjected to external perturbations. We consider networks whose nodes have two internal states labeled 0 and 1. We let N_0 nodes be frozen in state 0, N_1 in state 1, and the remaining nodes change by adopting the state of a connected node with a fixed probability per time step. The frozen nodes can be interpreted as external perturbations to the subnetwork of free nodes. Analytically extending N_0 and N_1 to be smaller than 1 enables modeling the case of weak coupling. We solve the dynamical equations exactly for fully connected networks, obtaining the equilibrium distribution, transition probabilities between any two states and the characteristic time to equilibration. Our exact results are excellent approximations for other topologies, including random, regular lattice, scale-free and small world networks, when the numbers of fixed nodes are adjusted to take account of the effect of topology on coupling to the environment. This model can describe a variety of complex systems, from magnetic spins to social networks to population genetics, and was recently applied as a framework for early warning signals for real-world self-organized economic market crises.

PACS numbers: 89.75.-k,05.50.+q,05.45.Xt

I. INTRODUCTION

Networks have become a standard model for a wealth of complex systems, from physics to social sciences to biology [1–6]. A large body of work has investigated topological properties [1, 3, 4, 7] including changes due to node removal [8–10]. The *raison d'être*, though, of complex network studies is to understand the relationship between structure and dynamics [11] - from disease spreading and social influence [12–16] to search[17] and time dependent networks [18, 19]. Yet, dynamic response of networks under external perturbations has been less thoroughly investigated [7, 20, 21]. In this paper we consider a simple dynamical process as a general framework for the dynamic response of a network to an external environment. We are interested in the dynamics of propagation of perturbations and in the process of equilibration of the network as promoted by the external environment. The environment is initially treated as a small portion of the network itself and is later generalized as an external system.

We take the voter model [22] as our basic dynamical system. It consists of voters, represented by nodes on a network, having only two possible opinions, 0 or 1. Whereas each voter may change his mind by randomly adopting the opinion of a connected neighbor, the existence of voters that will not change their minds (zealots) may be seen as external perturbations to the subnetwork of undecided voters. This problem has been studied with a single zealot in regular lattices [23] and with arbitrary number of zealots in fully connected networks where analytic solutions were obtained in the limit where the number of voters go to infinity [24].

Here we obtain complete and exact results in terms of hypergeometric generating functions for the simplest case of fully connected networks and arbitrary number of voters and zealots. We find a nontrivial dynamic behavior that can be divided into two regimes for small and large perturbations. For large perturbations the environmental influence extends into the system with a characteristic magnitude and a distribution which, in the thermodynamic limit, becomes a Gaussian. For small perturbations, on the other hand, the probability distribution of internal states displays a power law behavior peaked on the unperturbed solutions. The boundary between these two regimes is characterized by a uniform distribution where all states are equally likely. The time scale of equilibration is determined and is small for large perturbations and diverges inversely with the strength of the perturbation for small perturbations. Remarkably, simulations show that the same results apply for diverse network topologies, though the constants in the distributions become renormalized based upon topological properties. This makes our analytical results useful for a wide variety of networked systems, from magnetic spins and population genetics to social networks and opinion dynamics [28], and was also applied as a framework for early warning signals of real-world sef-organized economic and market crises [29]. We note that a partial account of these results are available in [30].

II. MODEL

Consider a network with $N + N_0 + N_1$ nodes. Each node has an internal state which can take only the values 0 or 1. We assume that N nodes are free to change their internal state according to the rule described below, while N_0 nodes are frozen in state 0 and N_1 nodes in state 1. At each time step a random free node is selected and its state is updated as follows: with probability p the state remains the same; with probability $1 - p$ the node copies the state of one of its connected neighbors, chosen at random.

This system can model a number of situations. An example is the Ising model, where our dynamics are equivalent to Glauber dynamics [25] for small external magnetic fields (h) and all temperatures (T) including the phase transition regime, for uniform connectivity lattices in the thermodynamic limit. The Ising model parameters are $J/kT \to 1/(z + N_0 + N_1)$ and $h/J \to (N_1 - N_0)$, where z is the number of nearest neighbors and J the nearest-neighbor interaction strength. Relevant network structures include crystalline 3-D lattices and random networks for amorphous spin-glasses; fully connected networks correspond to long range interactions or the mean field approximation. The system can also model an election with two candidates [26, 27] where some of the voters have a fixed opinion while the rest change their intention according to the opinion of others. Another application is to epidemics that spread upon contact between infected nodes (e.g., individuals or computers), a case for which we would set $N_0 = 0$ to study spreading dynamics. Finally, the model can represent an evolving population of sexually reproducing (haploid) organisms where the internal state represents one of two alleles of a gene [31, 32]. Taking $p = 1/2$, the update of a node mimics the mating of two individuals, with one parent being replaced by the offspring, which can receive the allele of either the mother or the father with 50% probability. Since a free node can also copy the state of a frozen node, the ratios $N_0/(N+N_0+N_1-1)$ and $N_1/(N+N_0+N_1-1)$ can be interpreted as mutation rates. If $N_0 \neq N_1$, the dynamics can represent selection towards one of the alleles or mutational bias.

III. FULLY CONNECTED NETWORKS

For the special case of a fully connected network the nodes are indistinguishable and the state of the network is fully specified by the number of nodes with internal state 1 [33]. Therefore, there are only $N+1$ global states, which we denote σ_k, $k=0,1,...,N$. The state σ_k has k free nodes in state 1 and $N-k$ free nodes in state 0. If $P_t(m)$ is the probability of finding the network in the state σ_m at the time t, then $P_{t+1}(m)$ can depend only on $P_t(m)$, $P_t(m+1)$ and $P_t(m-1)$. The probabilities $P_t(m)$ define a vector of $N+1$ components $\mathbf{P}_t$. The dynamics is described by the equation

$$\begin{aligned} P_{t+1}(m) \; &= P_t(m)\left\{p + \frac{(1-p)}{N(N+N_0+N_1-1)}\left[m(m+N_1-1)+(N-m)(N+N_0-m-1)\right]\right\} + \\ &P_t(m-1)\frac{(1-p)}{N(N+N_0+N_1-1)}(m+N_1-1)(N-m+1) + \\ &P_t(m+1)\frac{(1-p)}{N(N+N_0+N_1-1)}(m+1)(N+N_0-m-1)\;. \end{aligned}$$

The term inside the first brackets gives the probability that the state σ_m does not change in that time step and is divided into two contributions: the probability p that the node does not change plus the probability $1-p$ that the node does change but copies another node in the same state. In the latter case, the state of the node is 1 with probability m/N, and it may copy a different node in the same state with probability $(m-1+N_1)/(N+N_0+N_1-1)$. Also, if the state of the selected node is 0, which has probability $(N-m)/N$, it may copy another node in state 0 with probability $(N-m-1+N_0)/(N+N_0+N_1-1)$. The other terms are obtained similarly.

In terms of $\mathbf{P}_t$ this equation assumes the compact form

$$\mathbf{P}_{t+1} = \mathbf{U}\mathbf{P}_t \equiv \left(\mathbf{1} - \frac{(1-p)}{N(N+N_0+N_1-1)}\mathbf{A}\right)\mathbf{P}_t$$

where the *evolution matrix* $\mathbf{U}$, and also the auxiliary matrix $\mathbf{A}$, is tri-diagonal. The non-zero elements of $\mathbf{A}$ are independent of p and are given by

$$A_{m,m} = 2m(N-m) + N_1(N-m) + N_0 m$$
$$A_{m,m+1} = -(m+1)(N+N_0-m-1)$$
$$A_{m,m-1} = -(N-m+1)(N_1+m-1).$$

The transition probability from state σ_M to σ_L after a time t can be written as

$$P(L,t;M,0) = \sum_{r=0}^{N} b_{rM} a_{rL} \lambda_r^t \, . \tag{1}$$

where a_{rL} and b_{rM} are the components of the right and left r-th eigenvectors of the evolution matrix, $\mathbf{a}_r$ and $\mathbf{b}_r$, with $\mathbf{b}_r \cdot \mathbf{a}_r = \sum_{m=0}^{N} a_{rm} b_{rm} = 1$. Thus, the dynamical problem has been reduced to finding the right and left eigenvectors and the eigenvalues of $\mathbf{A}$.

It is easy to check by inspection of small matrices that the eigenvalues μ_r of $\mathbf{A}$ are given by

$$\mu_r = r(r-1+N_0+N_1)$$

so that the eigenvalues of $\mathbf{U}$ are

$$\lambda_r = 1 - \frac{(1-p)}{N(N+N_0+N_1-1)} \mu_r.$$

This implies that $0 \le p \le \lambda_r \le 1$. Because of Eq.(1), the unit eigenvalues completely determine the asymptotic behavior of the system.

The eigensystem $\mathbf{A}\mathbf{a}_r = \mu_r \mathbf{a}_r$ leads to the following recursion relation for the coefficients a_{rm}

$$\sum_{j=m-1}^{m+1} A_{mj}\, a_{rj} = \mu_r\, a_{rm} \tag{2}$$

with $a_{r,N+1} = a_{r,-1} \equiv 0$. To solve this equation we multiply the whole expression by x^m, sum over m and define the generating function $p_r(x) = \sum_{m=0}^{N} a_{rm} x^m$. Using relations such as $\sum_{m=0}^{N} m a_{rm} x^m = x p_r'$ and $\sum_{m=0}^{N} m a_{r,m+1} x^m = p_r' - p_r/x + a_0/x$, where the prime signifies differentiation with respect to x, we transform the recursion relation into the following differential equation for p_r:

$$\begin{aligned} &x(1-x)p_r'' + [(1-N-N_0) - (1+N_1-N)x]p_r' + \\ &[NN_1 - \mu_r/(1-x)]p_r = 0. \end{aligned} \tag{3}$$

To understand the asymptotic behavior of the system ($\mu_r = 0$) we have to consider two cases:

(a) If $N_0 = N_1 = 0$ then $\mu_r = 0$ leads to $r = 0$ or $r = 1$ [33]. In this case the differential equation simplifies to $xp_r'' + (1-N)p_r' = 0$, whose two independent solutions are $p_0(x) = 1$ and $p_1(x) = x^N$, corresponding to the all–nodes–0 or all–nodes–1 states respectively.

(b) If $N_0, N_1 \neq 0$ then $\mu_r = 0$ implies $r = 0$. In this case equation (3) is that of a hypergeometric function F and we find $p_0(x) = F(-N, N_1, 1-N-N_0, x)$, which is a finite polynomial with known coefficients a_{0m}. Normalizing this eigenvector, we obtain the probability of finding the network in state σ_m at large times:

$$\rho(m) = \mathcal{A}(N, N_0, N_1) \; \frac{(N_1+m-1)! \; (N+N_0-m-1)!}{(N-m)! \; m!} \tag{4}$$

where

$$\mathcal{A}(N, N_0, N_1) = \frac{N! \; (N_0+N_1-1)!}{(N+N_0+N_1-1)! \; (N_1-1)! \; (N_0-1)!}.$$

Because of the frozen nodes, the dynamics will never stabilize in any state, but will always move from one state to another, with mean occupation number $\bar{m} = NN_1/(N_0+N_1)$. The surprising feature of this solution is that for $N_0 = N_1 = 1$ we obtain $\rho(m) = 1/(N+1)$, for all values of N. Thus all states are equally likely and the system executes a random walk through the state space.

The dynamics at long times is dominated by the second largest eigenvector with eigenvalue λ_1. For large networks $\lambda_1^t \approx e^{-t/\tau}$ where

$$\tau = \frac{N(N+N_0+N_1-1)}{(1-p)(N_0+N_1)} \tag{5}$$

is the relaxation time. Equations (4) and (5) are important results of this paper.

We obtain a complete description of the dynamics by deriving all eigenvectors with $\mu_r \neq 0$. The differential equation for $p_r(x)$ can still be solved in terms of hypergeometric functions:

$$p_r(x) = \frac{F(1-r-N_0, 1-r-N-N_0-N_1, 1-N-N_0, x)}{(1-x)^{r-1+N_0+N_1}}. \tag{6}$$

Expanding the numerator and denominator in Taylor series gives the coefficients a_{rm}. Although they can easily be written down explicitly, we will omit their expressions, since they are not particularly illuminating.

The calculation of the left eigenvectors proceeds similarly. Defining the generating function $q_r(x) = \sum_{m=1-N_1}^{N+N_0+N_1} b_{rm} x^m$ we obtain a differential equation for q_r whose solution is

$$q_r(x) = \frac{x^{1-N_1} \; F(1-r-N_1, 1-r-N-N_0-N_1, 1-N-N_1, x)}{(1-x)^{r+1}}. \tag{7}$$

If $N_0 = N_1 = 0$ this solution is not valid for $r = 0$ or $r = 1$, since the matrix $\mathbf{A}^T$ becomes singular. In this case the two (unnormalized) left eigenvectors are given by $b_{0,m} = 1$ and $b_{1,m} = N - 2m$. For all other cases the solution is obtained from the expansion of $q_r(x)$ in power series. Once again we shall not write down the expansion coefficients explicitly. Equations (6) and (7) complete the dynamical solution of the problem.

In the thermodynamic limit $N \to \infty$ we can define continuous variables $x = m/N$, $n_0 = N_0/N$ and $n_1 = N_1/N$ and approximate the asymptotic distribution by a Gaussian $\rho(x) = \rho_0 \exp\left[-(x - x_0)^2/2\delta^2\right]$ with $x_0 = n_1/(n_0 + n_1)$, $\rho_0 = 1/\sqrt{2\pi\delta^2}$ and

$$\delta = \left[\frac{n_0 n_1 (1 + n_0 + n_1)}{N(n_0 + n_1)^3}\right]^{1/2}. \tag{8}$$

In the limit where $n_0,\ n_1 >> 1$ the width depends only on the ratio $\alpha = n_0/n_1$ and is given by $\sqrt{\alpha/N}/(1 + \alpha)$. In particular, for $n_0 = n_1 >> 1$, the width tends to $1/(2\sqrt{N})$.

The problem we just solved can be generalized to treat an external reservoir weakly coupled to the network of N nodes. We note that the differential equations for the generating functions $p_r(x)$ and $q_r(x)$ remain perfectly well defined if N_0 and N_1 are real numbers. The solutions for the generating functions also remain the same, with the difference that factorials must be replaced by gamma functions. Since the numbers $N_0/(N + N_0 + N_1 - 1)$ and $N_1/(N + N_0 + N_1 - 1)$ represent the probabilities that a free node copies one of the frozen nodes, small values of N_0 and N_1 can be interpreted as representing a weak connection between the free nodes and an external system containing the frozen nodes. The external system can be thought of as a reservoir that affects the network but is not affected by it. Alternatively, we can suppose that there is a single node fixed at 0 that is on for only a fraction N_0 of the time and off for the fraction $1 - N_0$, and similarly for a single node fixed at 1.

Figure 1 shows examples of the distribution $\rho(m)$ for a network with $N = 100$ and various values of N_0 and N_1. Numerical simulations displaying similar results have been described in [34]. For $N_0 = N_1$ a phase transition between disordered and ordered states occurs in the limit $N \to \infty$ at $N_0 = N_1 = 1$: for $N_0 = N_1 >> 1$ about half the nodes are in state 0 and half in state 1, similarly to a magnetic material at high temperatures. For $N_0 = N_1 << 1$, on the other hand, the distribution peaks at all nodes 0 or all nodes 1, similar to a magnetized state at low temperatures.

Figure 2 shows an example of the time evolution of the probability density for a fully

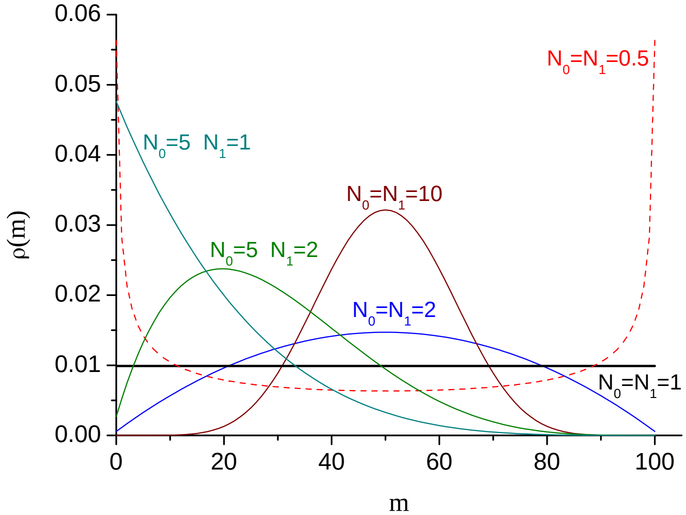

FIG. 1. Asymptotic probability distribution for a network with $N = 100$ nodes and several values of N_0 and N_1.

connected network compared to numerical simulations. The evolution from the initial to the asymptotic time-independent distribution is the analog of an equilibration process promoted by the external system.

For small values of N_0 and N_1 ($<< 1/\ln N$), we can obtain a simplified expression for $\rho(m)$:

$$\rho(m) \approx \frac{N_1 N_0}{N_0 + N_1} \left[\frac{1 - N_1 \ln N}{m^{1-N_1}} + \frac{1 - N_0 \ln N}{(N-m)^{1-N_0}} \right]. \tag{9}$$

Thus $\rho(m)$ displays a power law behavior on both ends of the curve: $1/m$ for m close to 0 and $1/(N-m)$ for m close to N (see, for instance, the curve with $N_0 = N_1 = 0.5$ in Fig. 1). Since the relaxation time τ is proportional to $1/(N_0 + N_1)$, the equilibration process becomes very slow in this limit.

IV. OTHER TOPOLOGIES

Fully connected networks are rarely found in nature. On the contrary, most networks representing social, biological or physical systems have complex topologies where the distribution of links is highly inhomogeneous. For these networks, which are not fully connected, the effect of the frozen nodes is amplified and can be quantified as follows: the

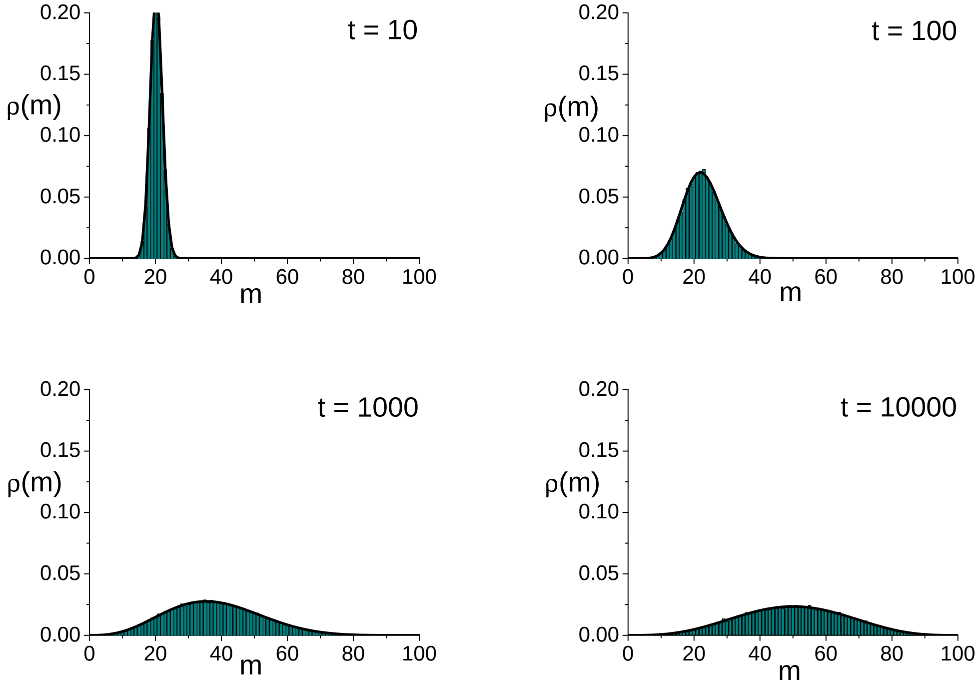

FIG. 2. Time evolution of the probability distribution P_t for a network with $N = 100$ and $N_0 = N_1 = 5$. The histograms show the average over 50,000 actual realizations of the dynamics and the solid line shows the analytical result.

probability that a free node copies a frozen node is $P_i = (N_0 + N_1)/(N_0 + N_1 + k_i)$ where k_i is the degree of the node. For fully connected networks $k_i = N - 1$ and we obtain $P_{FC} \equiv (N_0 + N_1)/(N_0 + N_1 + N - 1)$. For general networks an average value P_{av} can be calculated by replacing k_i by the average degree $k_{av} = 1/N \sum_i k_i$. We can then define effective numbers of frozen nodes, N_{0ef} and N_{1ef}, as being the values of N_0 and N_1 in P_{FC} for which $P_{av} \equiv P_{FC}$. This leads to

$$N_{0ef} = f N_0, \qquad N_{1ef} = f N_1 \tag{10}$$

where $f = (N-1)/k_{av}$. For well behaved distributions, corrections involving higher moments can be obtained by integrating P_i times the degree distribution and expanding around k_{av}.

Figure 3 shows examples of the equilibrium distribution attained by networks with different topologies. Panel (a) shows a random network with connection probability between nodes of $pc = 0.3$ (nodes have 30 connections each on the average). The theoretical result is given by Eq. (4) but for $N_{0ef} = N_{1ef} = 17 \approx N_0/pc$. The larger effective values of N_0

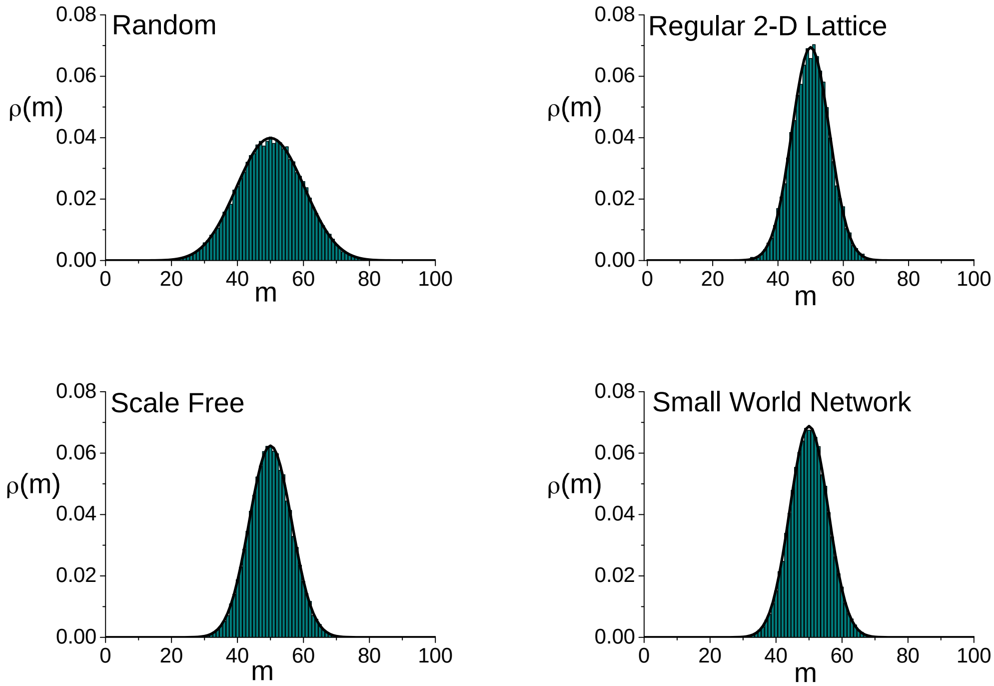

FIG. 3. Asymptotic probability distribution for networks with different topologies. In all cases $N = 100$, $N_0 = N_1 = 5$, $t = 10,000$, and the number of realizations is $50,000$. The theoretical curve is drawn with effective numbers of frozen nodes N_{0ef} and N_{1ef}: (a) random network $N_{0ef} = N_{1ef} = 17$; (b) regular 2-D lattice $N_{0ef} = N_{1ef} = 150$; (c) scale-free $N_{0ef} = N_{1ef} = 80$; (d) small world network $N_{0ef} = N_{1ef} = 143$.

and N_1 in this case are easy to understand: the weaker propagation of the perturbations resulting from the smaller connectivity is compensated by an increase in the effective size of the perturbation. Panel (b) shows the probability distribution for a 2-D regular lattice with 10×10 nodes. This time the theoretical result fits the curve only if $N_{ef0} = N_{ef1} = 150$ which, once again is of the order of $99N_0/4$, where 99 is the number of neighbors in the fully connected case and 4 the number of neighbors in the regular lattice. For a scale-free network (panel (c)) grown from an initial cluster of 6 nodes adding nodes with 3 connections each following the preferential attachment rule [1], the effective values of N_0 and N_1 are 80. Since the average number of connections per node in this network is close to 3, the linear rule applied for the random and regular networks would result in $N_{ef0} = N_{ef1} = 165$. Thus the scale-free topology plays an important role in propagating the perturbations more effectively than in regular networks. Finally, panel (d) shows a small world version of the regular lattice [35], where 30 connections were randomly re-connected, creating shortcuts between otherwise distant nodes. The average number of connections per node is the same as in the regular lattice, but the effective size of the perturbations decreases to $N_{ef0} = N_{ef1} = 143$, since the shortcuts promote faster propagation.

The fit of equilibrium distributions by effective values presented in Fig.3 holds for unequal values of N_0 and N_1. These effective values can also be used to describe the dynamics quite accurately, as long as the initial state σ_i is constructed by randomly assigning i nodes with state 1 and $N-i$ with state 0. However, if the initial state is specially prepared, for instance, assigning the value 1 to the most connected nodes of a scale-free network, the short time dynamics can be quite different from the theoretical prediction.

V. CONCLUSION

In this paper we considered a simple dynamical process on networks where binary states are assigned to nodes. The state of the nodes may change stochastically depending on the state of their neighbors and on external perturbations represented by the frozen nodes. We have solved the problem for fully connected networks and provided approximate formulas for other topologies by rescaling the perturbation.

The expression (4) gives the probability distribution of nodes in state 1 as a function of the three parameters N, N_0 and N_1. It can also be written in terms of the total perturbation $N_1 + N_0$ and the bias towards one of the states, $N_1 - N_0$, similar to temperature and magnetic field in the Ising model. Applications of this expression in population genetics [32] and financial markets [29] have been pointed out recently.

M.A.M.A. and D.D.C. acknowledge financial support from CNPq and FAPESP. I.R.E. was supported by National Science Foundation grant CHE-1362477.

---

[1] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).

[2] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez and D.-U. Hwang, Phys. Rep. **424**, 175 (2006).

[3] M.E.J. Newman, SIAM Review **45**, 176 (2003).

[4] M.E. J. Newman, *Networks: An Introduction*, Oxford University Press, 2010.

[5] O. Sporns, *Networks of the Brain* The MIT Press, Cambridge, 2011.

[6] *Adaptive Networks. Theory, Models and Applications (Understanding Complex Systems).* T. Gross and H. Sayama, (eds.) (Springer-Verlag: Berlin, 2009)

[7] Y. Bar-Yam and I. Epstein, PNAS **101**, 4341 (2004).

[8] R. Albert, H. Jeong, and A.-L. Barab?si, Nature London **406**, 378 (2000).

[9] R. Cohen, K. Erez, D. ben-Avraham, and S. Havlin, Phys. Rev. Lett. **85**, 4626 (2000).

[10] Buldyrev, S. V., Parshani, R., Paul, G., Stanley, H. E., and Havlin, S. Nature **464**, 1025-1028 (2010).

[11] Barrat, A., Barthelemy, M., and Vespignani, A. (2008). *Dynamical processes on complex networks* (Vol. 1). Cambridge: Cambridge University Press.

[12] R. Pastor-Satorras and A. Vespignani, Phys. Rev. Lett. **86**, 3200 (2001).

[13] M. Barahona and L.M. Pecora, Phys. Rev. Lett. **89**, 054101 (2002).

[14] T. Nishikawa, A.E. Motter, Y.-C. Lai, and F.C. Hoppensteadt, Phys. Rev. Lett. **91**, 014101 (2003).

[15] Y. Moreno, M. Nekovee and A.F. Pacheco, Phys. Rev. E **69**, 066130 (2004).

[16] M.F. Laguna, A. Guillermo, and D. H. Zanette, Physica A **329**, 459 (2003).

[17] R. Guimera et al., Phys. Rev. Lett. **89**, 248701 (2002).

[18] D. Braha and Y. BarYam, Complexity, **12**(2), 59 (2006).

[19] S.A. Hill and D. Braha, Phys. Rev. E **82**(4), 046105 (2010).

[20] X. Wang, Y.-C. Lai, and C. H. Lai, Phys. Rev. E **74**, 066104 (2006).

[21] E.J. Lee, K.-I. Goh, B. Kahng, and D. Kim, Phys. Rev. E **71**, 056108 (2005).

[22] T. M. Liggett, *Interacting Particles Systems* (Springer, New York, 1985).

[23] M. Mobilia, Phys. Rev. Lett. **91**, 028701 (2003).

[24] M. Mobilia, A. Petersen, S. Redner, J. Stat. Mech. P08029 (2007) doi:10.1088/1742-5468/2007/08/P08029.

[25] R.J. Glauber, J. Math. Phys. **4**, 294 (1963)

[26] D. Vilone and C. Castellano, Phys. Rev. E **69**, 016109 (2004).

[27] V. Sood and S. Redner, Phys. Rev. Lett. **94**,178701 (2005).

[28] E. Yildiz, A. Ozdaglar, D. Acemoglu, A. Saberi and A. Scaglione, ACM Trans. Econ. Comp.

1, 4, Article 19 (2013). DOI:http://dx.doi.org/10.1145/2538508

[29] D. Harmon, M.A.M. de Aguiar, D. Chinellato, D. Braha, I.R. Epstein and Y. Bar-Yam, eprint, arXiv:1102.2620 [q-fin.ST] (2011).

[30] D. D. Chinellato, M. A. M. de Aguiar, I. R. Epstein, D. Braha, and Y. Bar-Yam, e-print, arXiv:0705.4607v2 [nlin.SI] (2007).

[31] P.A.P. Moran, Proc. Cambridge Philos. Soc. **54**, 60 (1958).

[32] M.A.M. de Aguiar and Y. Bar-Yam, Phys. Rev. E **84** 031901 (2011).

[33] M.A.M. de Aguiar, I.R. Epstein and Y. Bar-Yam, Phys. Rev. E **72**, 067102 (2005).

[34] N. Boccara, eprint, arXiv:0704.1790 [nlin.AO] (2007).

[35] D.J. Watts and S.H. Strogatz, Nature **393**, 440 (1998).