

Floating bubbles in one-parameter Hamiltonian systems

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In one-parameter families of Hamiltonians with two degrees of freedom, some periodic orbits may form isolated families, also called "floating bubbles." In this paper we study, numerically, two examples of floating bubbles and show that the lack of symmetry in a plot of the trace of its monodromy matrix versus the energy implies the connection of these families to other families of periodic orbits through period- n -tupling bifurcations.

It is well known that in nonintegrable Hamiltonian systems with two degrees of freedom the periodic solutions constitute one-parameter families. The basic rules for the occurrence of bifurcations (period- n -tupling) in these families of periodic orbits are well established (see, for example, Refs. 1 and 2). The presence of symmetries in the Hamiltonian (time reversal, reflexion symmetries) enriches the bifurcation portrait;^{3,4} nevertheless, we know where bifurcations occur and the kind of bifurcations that may occur if we follow any family of periodic orbits.

In the work by Aguiar *et al.*³ a large number of families of periodic orbits were numerically obtained for the nonintegrable Hamiltonian

$$H(x, p_x, y, p_y) = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{x^2}{2} + \frac{3y^2}{2} + \frac{x^4}{12} - \alpha x^2 y, \quad (1)$$

in the case when $\alpha = 1$. Extensive numerical calculations of periodic orbits were also done for other nonintegrable Hamiltonians.⁵ The numerical data were displayed in energy versus period ($\epsilon\tau$) plots, where a family of periodic orbits is represented by a continuous curve. The Hamiltonians studied were all harmonic at very low energies. We found that the *central families* of periodic orbits (the families tending to harmonic oscillations as their amplitudes go to zero) are interconnected through period- n -tupling bifurcations. When a single family is self-connected by a bifurcating orbit, we say that a *bubble* has been formed, due to the topology of these curves on the $\epsilon\tau$ plot. This terminology was first invented by Contopoulos⁶ although he used a different representation for plotting the periodic-orbit families.

Besides these central families that are interconnected, we found families of periodic orbits for which the corresponding $\epsilon\tau$ plots form isolated closed cycles that do not connect to any family. These families correspond to the *floating bubbles* in the notation used by Contopoulos.⁶ In that work a possible mechanism for the formation of these floating bubbles was presented: varying α [a parameter in the Hamiltonian, usually the strength of the nonintegrable perturbation term in Eq. (1)], they may appear by detaching from another family when the points

that join that family coalesce. An example was given there in the case of a rotating Hamiltonian (used to model a rotating barred galaxy).

Another possibility for the appearance of floating bubbles was also given by Contopoulos⁶ and Contopoulos and Papayannopoulos;⁷ as α is varied, they simply pop up at some critical value already isolated from other families (see Figs. 1 and 2).

In this paper we give other numerical examples of floating bubbles found in the Hamiltonian (1) and we focus our attention on the global topological properties of these families in the $\epsilon\tau$ plot and on their stability properties.

In order to study the formation of isolated families, we made a numerical study of the evolution, as α is decreased, of two isolated families labeled E and F in Aguiar *et al.*³ Both of them are families of time-reversal and reflexion symmetric periodic trajectories. We found that, as α is decreased, for both E and F families, the corresponding closed cycles representing them in the $\epsilon\tau$ plot simply shrink (Figs. 1 and 2) and are finally reduced to a single point. This happens at different values of α for each family: $0.7621 < \alpha < 0.7622$ for the E family while $0.7357 < \alpha < 0.7358$ for the F family. These two families are generated, therefore, according to the second mechanism described above.

Notice that these values of α are not miscalculations of the critical value $\alpha_c = 1/\sqrt{2}$ at which the two saddle points of the potential function are generated. Actually, they have to be greater than α_c , for both E and F go back and forth along the corresponding valley as they are observed to do.

The two families of orbits studied here present some properties that seem quite general of floating bubbles. One of these properties is the shape of a figure eight in the $\epsilon\tau$ plot (see Aguiar *et al.* for details). Another interesting feature is the presence of two stable regions at the bottom and top of the figure eight. These two regions can be shown to always exist by standard bifurcation theory. As we go through these stable regions, an infinite number of bifurcations occur and then two possibilities arise: Either the bifurcated families of orbits are "born"

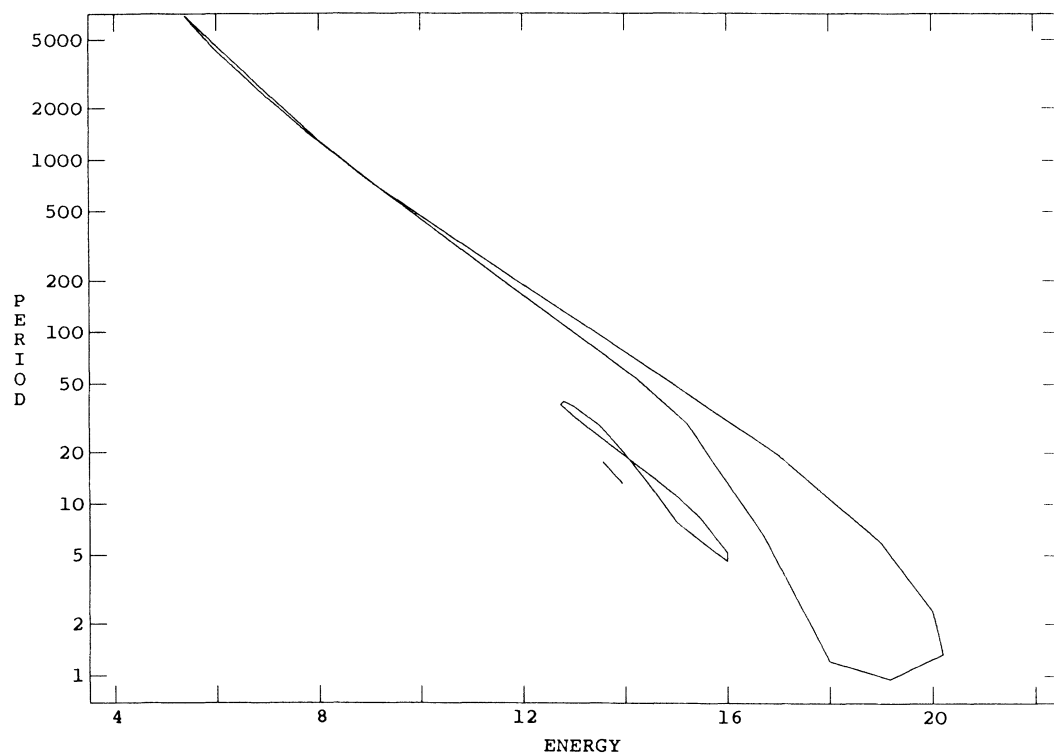


FIG. 1. $\varepsilon\tau$ plot of family E for $\alpha = 1.000, 0.800$, and 0.763 . As α decreases the figure-eight-shaped loop shrinks to a point.

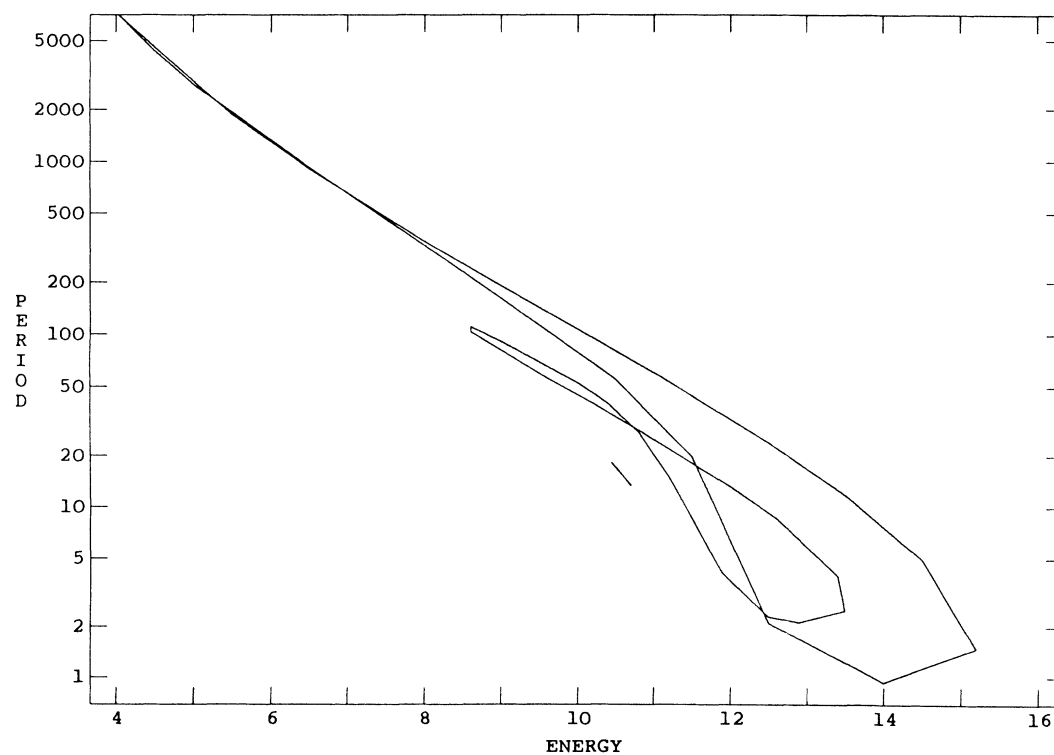


FIG. 2. $\varepsilon\tau$ plot of family F for $\alpha = 1.000, 0.855$, and 0.737 . Like family E , the cycle is not connected to any other family for any value of α .

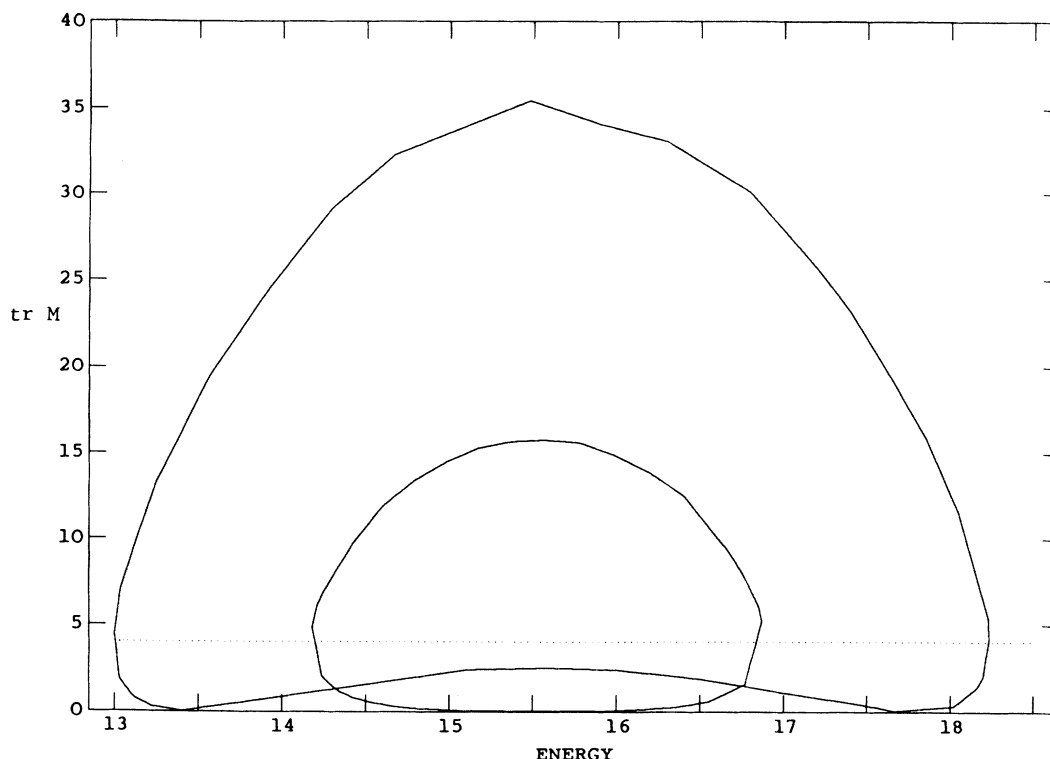


FIG. 3. $\text{tr} \underline{M} \epsilon$ for the F family for $\alpha=0.7370$ and 0.7361 (smaller loop). For two-dimensional systems the periodic orbits will be stable (except for a set of zero measure) if $0 \leq \text{tr} \underline{M} \leq 4$ and unstable otherwise. The dotted line is $\text{tr} \underline{M} = 4$.

at the bottom and “die” at the top (and in this case the bubble would be indeed isolated) or some of them may connect to other families of periodic orbits. Since it is virtually impossible to follow all these bifurcations to see what happens, we made a plot of the trace of the monodromy matrix \underline{M} versus the energy ϵ for one of the families studied, as shown in Fig. 3. We observe that as the parameter α decreases, the two stable regions (where $0 < \text{tr} \underline{M} < 4$) get together in a very symmetric way, showing that bifurcations with the same rotation number, from the two regions, coalesce at a certain critical value.

If the plot was not symmetric, orbits bifurcated from the top stability region and from the bottom stability region would disappear at different values of α and, therefore, the bubble would certainly be connected to another family of periodic orbits which would be linked to the other end of the bifurcated orbit which has disappeared.

The symmetry of the plot $\text{tr} \underline{M} \epsilon$, however, does not guarantee that the bubble is isolated, since a pair of bifurcated orbits from the top and from the bottom regions of the figure eight, with the same rotation number, may ei-

ther belong to the same family that disappears at the critical value of α , or constitute different families connected to other ones that get together and detach from the bubble at the critical value of α , realizing Contopoulos’ first mechanism of floating-bubble generation.

To summarize, although it is very difficult to show whether a floating bubble is connected to other families by its period- n -tupling bifurcated orbits, the symmetry properties of its monodromy matrix as a function of the energy may decide the question if $\text{tr} \underline{M} \epsilon$ is asymmetric. Therefore symmetry of the $\text{tr} \underline{M} \epsilon$ plot is a necessary but not sufficient condition for the isolation of floating bubbles.

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