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Classical dissipation and asymptotic equilibrium via interaction with chaotic systems

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Abstract

We study the energy flow between a one-dimensional oscillator and a chaotic system with two degrees of freedom in the weak coupling limit. The oscillator's observables are averaged over an initially microcanonical ensemble of trajectories of the chaotic system, which plays the role of an environment for the oscillator. We show numerically that the oscillator's average energy exhibits irreversible dynamics and 'thermal' equilibrium at long times. We use linear response theory to describe the dynamics at short times and we derive a condition for the absorption or dissipation of energy by the oscillator from the chaotic system. The equilibrium properties at long times, including the average equilibrium energies and the energy distributions, are explained with the help of statistical arguments. We also check that the concept of temperature defined in terms of the 'volume entropy' agrees very well with these energy distributions.
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1. Introduction

Low-dimensional chaotic systems can, under appropriate circumstances, play the role of thermodynamical heat baths [1–7]. If a slow system with few degrees of freedom is weakly coupled to a fast chaotic system, the slow system's average trajectory can dissipate energy into the chaotic one at short times.

One of the initial motivations for the consideration of low-dimensional chaotic dynamics as environments for macroscopic systems was the work by Brown et al. [8] on the ergodic adiabatic invariant. For a chaotic Hamiltonian system with a slowly varying parameter, the volume of the energy shell is the only adiabatic invariant. Brown et al. showed that the first order correction to this invariant has a diffusive temporal behavior. Besides, if such time varying parameter is thought of as a second system coupled to the chaotic one, then this diffusive correction of the adiabatic invariant would lead to a dissipative force on the slow system [1].

This problem was again reformulated by Berry and Robbins [2] in terms of a system of interest interacting with an environment, or 'thermal bath'. The average force acting on the system of interest due to the coupling with the chaotic system can be calculated in adiabatic approximation. The lowest order part of this force was

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shown to be the Born–Oppenheimer force and the next order to be a geometric magnetism type of force plus a deterministic friction—a force proportional to the slow system's velocity. Friction is therefore generated in the context of small systems, without the need for the thermodynamical limit, and the chaotic nature of the motion becomes the essential ingredient. Indeed, typical correlation functions of chaotic systems decay exponentially with the time, as opposed to the quasi-periodic behavior observed in regular systems. Jarzynski [4] showed later that coupling to a low-dimensional chaotic motion can also lead to a 'thermalization' of the system of interest, very much like the thermalization of a Brownian particle interacting with a large thermal bath. Finally, de Carvalho and de Aguiar [3] established a connection between de formalism developed by Caldeira and Leggett [9] for describing dissipation via coupling with a thermal bath and via coupling with a chaotic system.

In this paper we revisit this problem from the classical point of view and consider specific examples of two-dimensional chaotic systems coupled to a one-dimensional oscillator. Our main purpose here is to understand the energy flow between the system of interest, which we shall call 'the oscillator' and the chaotic system to which this oscillator is coupled. We study the energy flow at short times and the approach to equilibrium at long times. In order to consider the chaotic system as playing the role of an environment, we assume that the only information available from this system is its initial energy. For the oscillator this implies that only microcanonical averages of its observables (over the chaotic system variables) are accessible. Therefore, a typical numerical calculation corresponds to fix an initial condition for the oscillator and to evolve an ensemble of trajectories whose initial conditions for the chaotic variables are randomly selected at a fixed (chaotic system) energy shell.

For short times our numerical simulations show that the average energy of the oscillator may increase or decrease, absorbing energy from the chaotic system or dissipating energy into the chaotic system. The initial energies of both systems are what ultimately dictates which of the two possibilities actually occurs. In particular, there exists initial values of these energies such that no exchange occurs on the average. We use linear response theory to study the energy flow in the short time limit. We show that the average motion of the oscillator follows a Langevin type of equation with frequency-dependent friction and a quadratic correction to the oscillator potential, similar to the Born–Oppenheimer force that appears in the adiabatic theory. We also derive a simple condition for dissipation or absorption of energy by the oscillator involving the ratio of the initial energies of the systems. This theoretical prediction agrees well with our numerical calculations for short times, but it is not accurate to predict the long time behavior.

Our simulations show that, at long times, the average energy of both the oscillator and the chaotic system tend to an equilibrium. The value of these equilibrium energies depends once again on the initial conditions. The connection between asymptotic thermalization and initial conditions is well known for a Brownian particle. In that case, the increase or decrease of the average energy of the particle depends on its initial energy E_0 and on the temperature T of the thermal bath. The particle absorbs energy from the reservoir if $E_0 < k_B T$ and loses energy into it if $E_0 > k_B T$, thermalizing always at $k_B T$. Here we have a similar situation, with the increasing or decreasing of the average energy of the oscillator depending only on its initial energy and on the initial energy of the chaotic system. However, contrary to the case of the Brownian particle, the condition for equilibrium at long times is generally different from the condition of no energy exchange at short times. Despite the theoretical work of Jarzynski [4] on the long term thermalization of these systems, it is still not clear how the asymptotic states depend on the initial conditions of both sub-systems. The energy distribution of the sub-systems at equilibrium is also an important open issue. These distributions are not likely to be of the Boltzmann type, since the systems are small, and they may depend not only on the initial conditions but also on the density of states of the systems involved [10–12]. In this paper we shall derive the long time equilibrium conditions and energy distributions explicitly for two model systems. Finally, we use the definition of temperature proposed in Ref. [13] for small systems and check that it agrees completely with the statistical calculation in terms of the density of states and energy distributions.

We emphasize that our approach uses the linear response theory, and no explicit assumptions on adiabatic properties of the oscillator is required. Despite the difficulties involved in the calculation of the response function for microcanonical ensembles, the formulation of this problem in terms of linear response theory is of great interest for the study of its quantum analog [7]. We recall that the usual formulation of quantum dissipation [9] involves response functions and that the adiabatic approximation leads to frustrating results in

the quantum formulation [2]. The present work has some similarities with that of Ref. [3], which also considered the model of an oscillator coupled to a chaotic bath to study dissipation at short times. Here we study both the short and long time limits, showing that the coupling with the chaotic system may indeed lead to an equilibration very much like that caused by the coupling with a large thermal reservoir.

The outline of the paper is as follows: in Section 2 we describe our model systems and compute the correlation functions that are relevant for the calculation of the response functions. In Section 3 we review the classical theory of linear response and in Section 4 we apply the theory to our model. We calculate the response functions explicitly in the microcanonical ensemble. In Section 5 we compute the average energy and average equation of motion of the oscillator. We also derive a condition for the initial increase or decrease of the average energy of the oscillator for short times, and compare it with numerical results. In Section 6 we calculate the long time equilibrium values of the average energies using statistical arguments and compare them with numerical values. Finally in Section 7 we summarize our conclusions.

2. The model and first numerical results

Our model Hamiltonian consists of a one-dimensional harmonic oscillator $H_o(z)$ coupled to a chaotic system with two degrees of freedom $H_c(x, y)$:

$$H = H_o(z) + H_c(x, y) + V_I(x, z), \tag{1}$$

where

$$H_o(z) = \frac{p_z^2}{2m} + \frac{m\omega_0^2 z^2}{2}, \tag{2}$$

and

$$V_I(x, z) = \gamma x z. \tag{3}$$

For the chaotic Hamiltonian we consider two systems:

$$H_c(x, y) = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \left(y - \frac{x^2}{2} \right)^2 + 0.1 \frac{x^2}{2} \tag{4}$$

and

$$H_c(x, y) = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{x^2 y^2}{2} + a \frac{(x^4 + y^4)}{4}. \tag{5}$$

Hamiltonian (4) is known as the Nelson system (NS) [14] and we shall refer to (5) as the Quartic system (QS) [15]. The NS exhibits soft chaos and is fairly regular for $E_c \lesssim 0.05$, strongly chaotic for $E_c \gtrsim 0.3$ and mixed for intermediate values of the energy. The QS is invariant under a scaling transformation of the coordinates and the time, which implies that the dynamics on all energy shells are equivalent to each other. We shall explore this property later. The QS is integrable for $a = 1.0$, strongly chaotic for $a \lesssim 0.1$ and mixed for intermediate values of the parameter.

We want to investigate the situation in which the chaotic system plays the role of an external environment for the oscillator. Therefore, we assume that detailed information about the chaotic system is not available. If the environment were modelled by a heat bath, the only macroscopic relevant information would be its temperature. In the present case we assume that the only information available is the initial energy of the chaotic system. For the oscillator this implies that only microcanonical averages of its observables (over the chaotic system variables) are accessible.

In order to characterize the two chaotic systems Eqs. (4) and (5), we show in Fig. 1 the NS correlation functions $\langle x(0)x(t) \rangle_e$ and $\langle p_x(0)x(t) \rangle_e$, for $E_c = 0.38$ and, in Fig. 2, the QS correlation functions for $a = 0.1$ and $E_c = 5.0$. These functions play important roles in the linear response theory to be developed later. The brackets $\langle . \rangle_e$ stand for the average on the microcanonical ensemble of initial conditions. The correlation functions are obtained numerically with a fixed time step symplectic integration algorithm [16] applied to the isolated chaotic system. In our calculations we used time steps of the order of 10^{-4} for Figs. 1, 2, 5, 6 and 7
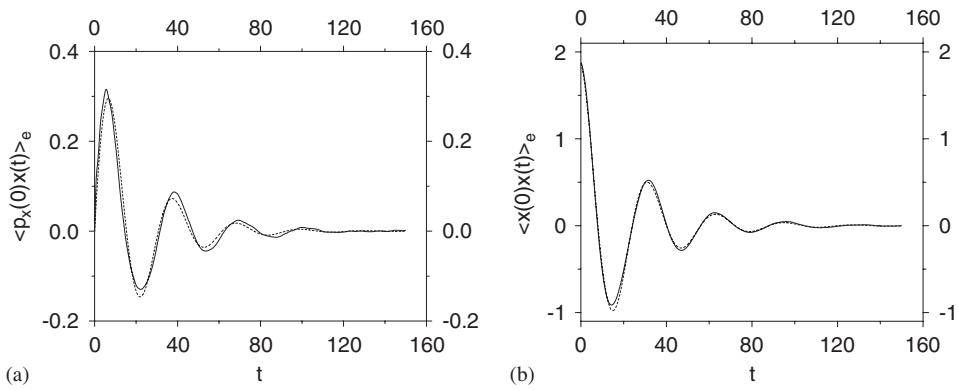
Fig. 1. Correlation functions for the NS for $E_c = 0.38$: (a) $\langle p_x(0)x(t)\rangle_e$; (b) $\langle x(0)x(t)\rangle_e$. The full line shows the numerical results and the dashed line shows the fitting. The averages were computed using 35 000 initial conditions.
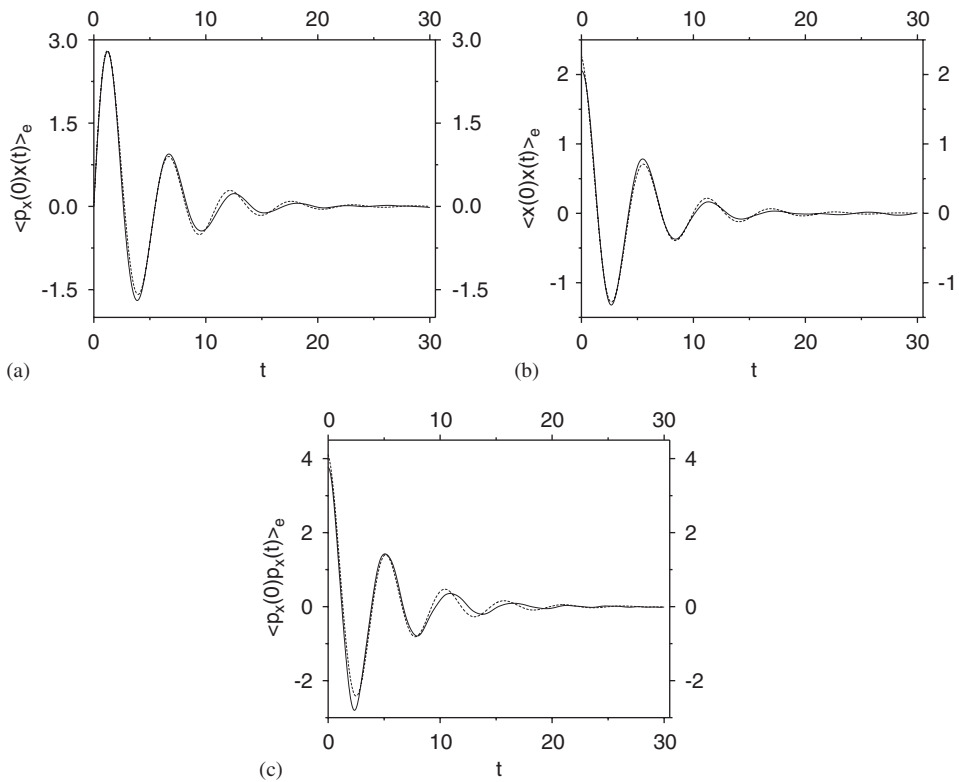

Fig. 2. Correlation functions for the QS for $E_c = 5.0$ and $a = 0.1$: (a) $\langle p_x(0)x(t)\rangle_e$; (b) $\langle x(0)x(t)\rangle_e$; and (c) $\langle p_x(0)p_x(t)\rangle_e$. The full line shows the numerical results and the dashed line shows the fitting. The averages were computed using 30 000 initial conditions.

(short times) and 10^{-2} for Figs. 3 and 4. The number of initial conditions in each case is indicated in the figure captions.

These correlation functions can be well fitted by the expressions

$$\langle x(0)x(t)\rangle_e = \sigma \mathrm{e}^{-\alpha t} \cos \omega t,$$
$$\langle p_x(0)x(t)\rangle_e = \mu \mathrm{e}^{-\beta t} \sin \Omega t,$$
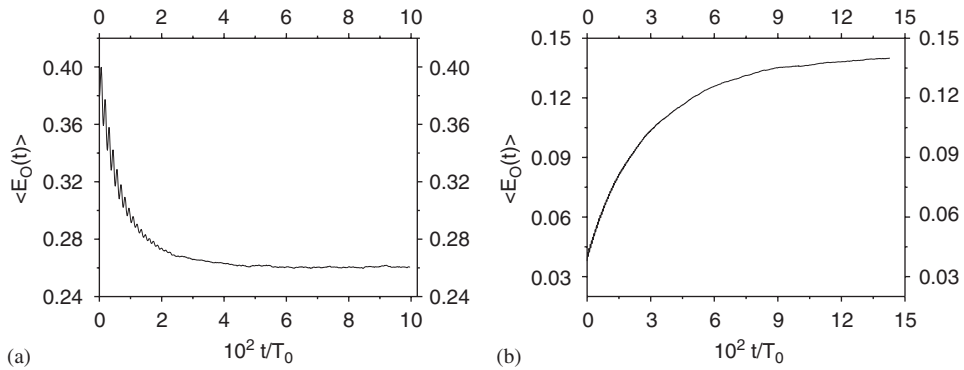$$\langle p_x(0)p_x(t)\rangle_e = \kappa \mathrm{e}^{-\lambda t} \cos \nu t \tag{6}$$

Fig. 3. Numerical results for $\langle E_o(t) \rangle$ with $E_c(0) = 0.38$ for NS: (a) $E_o(0)/E_c(0) = 1.0$ and (b) $E_o(0)/E_c(0) = 0.1$. The oscillator's parameters were set to $m = 200.0$, $\omega_0 = 0.005$ and the coupling constant to $\gamma = 0.006$. The oscillator's period is $T_0 \approx 1260$. We used 62 480 initial conditions.
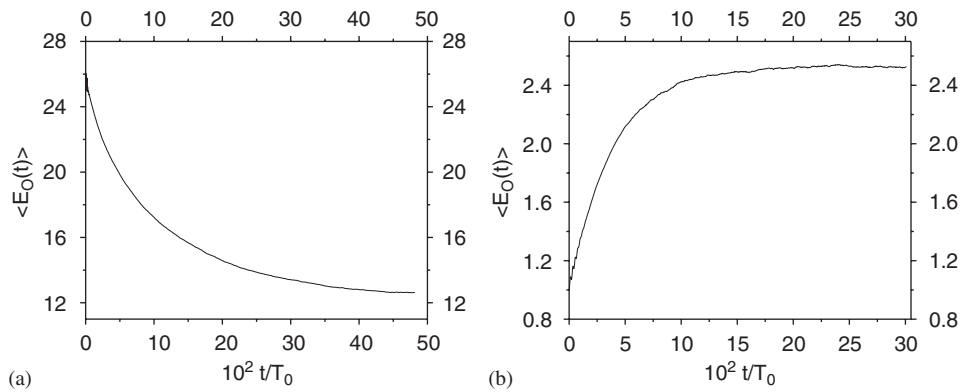

Fig. 4. Numerical results for $\langle E_o(t) \rangle$ with $E_c(0) = 5.0$ and $a = 0.1$ for the QS: (a) $E_o(0)/E_c(0) = 5.0$ and (b) $E_o(0)/E_c(0) = 0.2$. The oscillator's parameters were set to $m = 10.0$, $\omega_0 = 0.01$ and the coupling constant to $\gamma = 0.01$. The oscillator's period is $T_0 \approx 628$. We used 40 000 initial conditions.

(see Figs. 1 and 2), which have the proper parity of the corresponding correlation functions. For the NS the fitting furnishes the decay rates $\alpha = 0.0418$ and $\beta = 0.0456$, the amplitudes $\sigma = 1.865$ and $\mu = 0.409$ and the frequencies of oscillation $\omega = 0.1963$ and $\Omega = 0.2043$ with $\chi^2 \sim 10^{-4}$. For the QS we obtain $\alpha = 0.207$, $\beta = 0.208$ and $\lambda = 0.206$; $\sigma = 2.268$, $\mu = 3.67$ and $\kappa = 4.10$; and $\omega = 1.1027$, $\Omega = 1.1481$ and $v = 1.189$ with $\chi^2 \sim 10^{-3}$. This type of correlation function is typical of chaotic systems in general [2,17] and also occurs in non-isolated systems subjected to random noise, such as a Brownian particle. The source of the correlation loss is, however, different in each case. Notice that the numerically calculated exponents α, β and λ are all very similar and so are the frequencies ω, Ω and v.

When the coupling between the chaotic system and the oscillator is turned on, the overall conserved energy flows from one system to the other. The oscillator's energy, in particular, fluctuates as a function of the time for each specific trajectory. The oscillator's average energy is calculated by taking an ensemble of initial conditions uniformly distributed over the chaotic energy surface E_c. For the oscillator we fix only one initial condition, which we choose to be $z(0) = 0$ and $p_z(0) = \sqrt{2mE_o(0)}$. The microcanonical ensemble of initial conditions is propagated and, at each instant, H_o is calculated for each trajectory and its average value is computed. Figs. 3 and 4 show the oscillator's average energy $\langle E_o(t) \rangle$ as a function of the time for different values of $E_o(0)$. We observe that the average effect of the interaction with the chaotic system leads to a 'thermalization' of $\langle E_o(t) \rangle$. We also see that $E_o(0)$ plays an important role on the long term behavior of $\langle E_o(t) \rangle$.

As we shall see later using linear response theory, the ratio $E_o(0)/E_c(0)$ defines the initial rate of energy variation in time. Zooming the short time behavior of the curves in Figs. 3 and 4 (see Figs. 5 and 6) shows fast oscillations of $\langle E_o(t) \rangle$. As we shall see, part of these oscillations are due to a change in the effective potential acting of the oscillator, similar to the Born–Oppenheimer force of the adiabatic theory (see Eq. (48) in Section 5). Notice that the oscillator's equilibrium energy is different for each situation, which clearly distinguishes the small chaotic environment from an infinite thermal bath.

As a last remark we note that, since the environment has few degrees of freedom, the oscillator's motion for a single realization (single initial condition) exhibits large fluctuations with respect to the average.

3. Linear response theory

The calculation of averages such as $\langle E_o(t) \rangle$ involves the calculation of the microcanonical distribution function $\rho(q,p;t)$ whose initial value is $\rho(q,p;0) = \delta(H_c(q,p) - E_c(0))/\Sigma(E_c(0))$, with $\Sigma(E_c(0)) = \int dq\,dp\,\delta(H_c(q,p) - E_c(0))$. If the chaotic system were isolated, ρ would be an invariant distribution and $\rho(q,p;t) = \rho(q,p;0)$. The coupling, however, causes the value of $H_c(q(t),p(t))$ to fluctuate in time, distorting the energy surface $H_c = E_c(0)$ [18]. Linear response theory provides a way to calculate the first order corrections to this distribution in the limit of weak coupling [19,20].

Consider a Hamiltonian $H(q,p)$ perturbed by a term of the form

$$H_1 = -A(q,p)X(t), \tag{7}$$

where A is an arbitrary function of the coordinates and momenta and $X(t)$ is a function of the time. A generic distribution function $\rho(q,p)$ follows the Liouville equation

$$\frac{\partial \rho(t)}{\partial t} = i[L + L_1(t)]\rho(t), \tag{8}$$

where the Liouville operators are given by

$$iL\rho = \sum \left(\frac{\partial H}{\partial q}\frac{\partial \rho}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial \rho}{\partial q} \right) = \{H, \rho\},$$

$$iL_1\rho = \sum \left(\frac{\partial H_1(t)}{\partial q}\frac{\partial \rho}{\partial p} - \frac{\partial H_1(t)}{\partial p}\frac{\partial \rho}{\partial q} \right) = \{H_1(t), \rho\}, \tag{9}$$

where the summation is taken over the whole set of canonical variables.

The differential equation (8) has an integral representation of the form

$$\rho(t) = e^{i(t-t_0)L}\rho(t_0) + \int_{t_0}^{t} ds\, e^{i(t-s)L} iL_1(s)\rho(s). \tag{10}$$

Even if $\rho(t_0)$ is an equilibrium distribution of H (canonical or microcanonical, for example), it will not remain invariant if the perturbation H_1 is present. However, if the perturbation is small, we may expand Eq. (10) to first order in iL_1 to obtain

$$\rho(q,p;t) = \rho_e(q,p) + \int_{t_0}^{t} ds\, e^{i(t-s)L}\{H_1(s), \rho_e(q,p)\}, \tag{11}$$

where ρ_e denotes the initial equilibrium distribution and $\rho(q,p;t)$ is the time-dependent non-equilibrium distribution up to first order in the perturbation $H_1(t)$. Within this approximation, the average value of a general function $B(q,p)$ can be computed as

$$\langle B \rangle(t) = \int dq\,dp\,\rho_e(q,p)B(q,p) + \int_{t_0}^{t} ds\, \phi_{BA}(t-s)X(s), \tag{12}$$

where the *response function* $\phi_{BA}(t)$ is given by [19]

$$\phi_{BA}(t) = \int dq\,dp\,B(q,p)e^{itL}\{-A(q,p),\rho_e(q,p)\}$$
$$= \int dq\,dp\,\{\rho_e(q,p),A(q,p)\}B(q(t),p(t)), \tag{13}$$

with $q(t) = e^{itL}q$ and $p(t) = e^{itL}p$. Finally, expanding the Poisson brackets and integrating by parts we obtain

$$\phi_{BA}(t) = \int dq\,dp\,\rho_e(q,p)\{A(q,p),B(q(t),p(t))\}$$
$$= \langle\{A(q,p),B(q(t),p(t))\}\rangle_e. \tag{14}$$

The application of this theory to our model is as follows: we consider V_I as a perturbation on H_c, the coordinate z in V_I being the analog of $X(t)$ in (7). The initial microcanonical distribution for the chaotic system is then propagated according to Eq. (11). Averages of observables involving the chaotic system variables can also be calculated with the help of Eq. (12). In addition to the consideration that the oscillator acts as a perturbation to the chaotic system, we also consider the response of the chaotic system as a perturbation to the oscillator. This allows for the perturbative calculation of the oscillator's average trajectory and energy as well.

4. Application to the model

According to Eqs. (1)–(3), the equation of motion for the oscillator is given by

$$\ddot{z}(t) + \omega_0^2 z(t) = -\frac{\gamma}{m}x(t). \tag{15}$$

This equation can be solved explicitly in the form

$$z(t) = z_d(t) - \frac{\gamma}{m}\int_0^t ds\,\Gamma(t-s)x(s), \tag{16}$$

where z_d is the decoupled solution and $\Gamma(t-s) = \sin[\omega_0(t-s)]/\omega_0$.

Differentiating (16) with respect to t and multiplying by m we obtain

$$p_z(t) = p_{z_d}(t) - \gamma\int_0^t ds\,\chi(t-s)x(s) \tag{17}$$

with $\chi(t-s) = \cos[\omega_0(t-s)]$. Taking $z(0) = 0$ the decoupled solutions $z_d(t)$ and $p_{z_d}(t)$ become

$$z_d(t) = \frac{p_z(0)}{m\omega_0}\sin(\omega_0 t), \quad p_{z_d}(t) = p_z(0)\cos(\omega_0 t) \tag{18}$$

with $p_z(0) = \sqrt{2mE_o(0)}$. The oscillator's average energy $\langle E_o(t)\rangle$ can be computed as

$$\langle E_o(t)\rangle = \frac{\langle p_z^2(t)\rangle}{2m} + \frac{m\omega_0^2\langle z^2(t)\rangle}{2} \tag{19}$$

with

$$\langle p_z^2(t)\rangle = p_{z_d}^2(t) - 2\gamma p_{z_d}(t)\int_0^t ds\,\chi(t-s)\langle x(s)\rangle + \gamma^2\int_0^t ds\int_0^t du\,\chi(t-s)\chi(t-u)\langle x(s)x(u)\rangle \tag{20}$$

and

$$\langle z^2(t)\rangle = z_d^2(t) - \frac{2\gamma}{m}z_d(t)\int_0^t ds\,\Gamma(t-s)\langle x(s)\rangle + \frac{\gamma^2}{m^2}\int_0^t ds\int_0^t du\,\Gamma(t-s)\Gamma(t-u)\langle x(s)x(u)\rangle. \tag{21}$$

Notice that the quantity $\langle E_o(t)\rangle$ is very different from $E_o(\langle z\rangle,\langle p_z\rangle)$, which is the energy of the average trajectory, considered in Ref. [3].

Linear response theory now furnishes

$$\langle x(t) \rangle = \langle x(t) \rangle_e - \gamma \int_0^t ds\, \phi_{xx}(t-s) z(s), \tag{22}$$

where $\phi_{xx}(t) = \langle \{x(0), x(t)\} \rangle_e$ is the response function given by Eq. (14) with $A = B = x$. Since $H_c(-x) = H_c(x)$ for both chaotic systems given by Eqs. (4) and (5), $\langle x(t) \rangle_e = 0$. Substituting (22) into (20) and (21) and considering terms up to order γ^2 we obtain

$$\langle p_z^2(t) \rangle = p_{z_d}^2(t) + 2\gamma^2 \int_0^t ds\, \chi(t-s) \int_0^s du\, \phi_{xx}(s-u) z_d(u)$$
$$+ \gamma^2 \int_0^t ds \int_0^t du\, \chi(t-s)\chi(t-u)\langle x(s)x(u) \rangle_e, \tag{23}$$

$$\langle z^2(t) \rangle = z_d^2(t) + \frac{2\gamma^2}{m^2} \int_0^t ds\, \Gamma(t-s) \int_0^s du\, \phi_{xx}(s-u) z_d(u)$$
$$+ \frac{\gamma^2}{m^2} \int_0^t ds \int_0^t du\, \Gamma(t-s)\Gamma(t-u)\langle x(s)x(u) \rangle_e. \tag{24}$$

Eqs. (23) and (24) represent the average behavior of the harmonic oscillator when interacting with a generic x-symmetric system through the perturbation V_I. They show that all the influence of the interacting system is contained in the functions $\langle x(0)x(t) \rangle_e$ and $\phi_{xx}(t)$. If the interacting system is integrable, these functions exhibit quasi-periodic oscillations. For chaotic systems, on the other hand, they typically decay exponentially with time, leading to qualitatively distinct average results.

The response function is given by $\phi_{xx}(t) = \langle \{x(0), x(t)\} \rangle_e$ where $\{.\}$ is the Poisson bracket with respect to the initial conditions [19]. Since $p_x(0)$, $p_y(0)$, $x(0)$ and $y(0)$ are independent variables we integrate by parts and obtain

$$\phi_{xx}(t) = \int dV \{\rho_e, x(0)\} x(t) = -\int dV \frac{\partial \rho_e}{\partial p_x(0)} x(t), \tag{25}$$

where $dV = dx(0)\, dy(0)\, dp_x(0)\, dp_y(0)$,

$$\rho_e = \delta(H_c - E_c(0))/\Sigma(E_c(0)) \tag{26}$$

is the microcanonical distribution and

$$\Sigma(E) = \int \delta(H_c - E)\, dV \tag{27}$$

is a normalization factor. We obtain

$$\phi_{xx}(t) = -\frac{1}{\Sigma(E_c(0))} \int dV \frac{\partial \delta(H_c - E_c(0))}{\partial H_c} p_x(0) x(t). \tag{28}$$

This integral can be performed explicitly by changing to variables across the energy shell, H_c, and along the energy shell, ξ, θ and φ:

$$\phi_{xx}(t) = \int dH_c\, d\Theta \frac{\delta(H_c - E_c(0))}{\Sigma(E_c(0))} \frac{\partial}{\partial H_c} \left(J p_x(0) x(t) \right)$$
$$= \left\langle p_x(0)x(t) \frac{\partial J/\partial H_c}{J} \right\rangle_e + \left\langle \frac{\partial}{\partial H_c}(p_x(0)x(t)) \right\rangle_e, \tag{29}$$

where $\Theta = (\xi, \theta, \varphi)$, $J = J(H_c, \xi, \theta, \varphi)$ is the Jacobian of the transformation and the derivatives are computed at $E_c(0)$. Similar results for response functions calculated on the microcanonical ensemble can found in

Ref. [21]. For the NS, the explicit transformation is given by

$$x = \left(\frac{1}{0.05}\right)^{1/2} \sqrt{H_c} \cos \xi \cos \theta, \quad y = \frac{x^2}{2} + \sqrt{H_c} \sin \xi \cos \varphi,$$

$$p_x = \sqrt{2H_c} \cos \xi \sin \theta, \quad p_y = \sqrt{2H_c} \sin \xi \sin \varphi, \tag{30}$$

with $J = H_c f_N(\Theta)$. The response function simplifies to

$$\phi_{xx}(t) = \frac{\langle p_x(0)x(t)\rangle_e}{E_c(0)} + \left\langle \frac{\partial}{\partial H_c}(p_x(0)x(t)) \right\rangle_e. \tag{31}$$

For H_c close to $E_c(0)$ we assume that the amplitude of the correlation function $\langle p_x(0)x(t)\rangle_e$ varies linearly with H_c and we make the approximation

$$\left\langle \frac{\partial}{\partial H_c}(p_x(0)x(t)) \right\rangle_e \approx \frac{\langle p_x(0)x(t)\rangle_e}{E_c(0)}. \tag{32}$$

We obtain

$$\phi_{xx}(t) = \frac{2}{E_c(0)} \langle p_x(0)x(t)\rangle_e, \tag{33}$$

which is similar to the canonical response function [19].

For QS, the explicit transformation is given by

$$x^2 = \sqrt{\frac{2H_c}{\cos 2\theta}} \left[\frac{\cos \theta}{\sqrt{1+a}} + \frac{\sin \theta}{\sqrt{1-a}}\right] \sin \xi,$$

$$y^2 = \sqrt{\frac{2H_c}{\cos 2\theta}} \left[\frac{\cos \theta}{\sqrt{1+a}} - \frac{\sin \theta}{\sqrt{1-a}}\right] \sin \xi,$$

$$p_x = \sqrt{2H_c} \cos \varphi \cos \xi,$$

$$p_y = \sqrt{2H_c} \sin \varphi \cos \xi, \tag{34}$$

with $0 < \theta < \theta_0 < \pi/4$ and $\tan \theta_0 = \sqrt{(1-a)/(1+a)}$. The Jacobian is given as $J = H_c^{1/2} f_Q(\Theta)$ and the response function (29) simplifies to

$$\phi_{xx}(t) = \frac{\langle p_x(0)x(t)\rangle_e}{2E_c(0)} + \left\langle \frac{\partial}{\partial H_c}(p_x(0)x(t)) \right\rangle_e. \tag{35}$$

The last term of (35) can be calculated explicitly because the QS has an energy scaling dynamics [15] given by

$$p_x(t) = \left(\frac{E_c}{E_c'}\right)^{1/2} p_x'(t'), \quad x(t) = \left(\frac{E_c}{E_c'}\right)^{1/4} x'(t'), \quad t = \left(\frac{E_c}{E_c'}\right)^{-1/4} t', \tag{36}$$

where p_x', x' and t' represent the motion with the energy E_c' and p_x, x and t the motion with energy E_c.

Using (36) in (35), we obtain

$$\frac{\partial}{\partial H_c}(p_x(0)x(t)) = \frac{\partial}{\partial H_c} \left[\left(\frac{H_c}{E_c(0)}\right)^{3/4} p_x'(0)x'(t')\right]$$

$$= \frac{3}{4} \frac{1}{E_c^{3/4}(0)H_c^{1/4}} p_x'(0)x'(t') + \left(\frac{H_c}{E_c(0)}\right)^{3/4} p_x'(0) \frac{\partial x'(t')}{\partial H_c}, \tag{37}$$

where the last term is given by

$$
\begin{aligned}
\left(\frac{H_c}{E_c(0)}\right)^{3/4} p'_x(0)\frac{\partial x'(t')}{\partial H_c} &= \left(\frac{H_c}{E_c(0)}\right)^{3/4} p'_x(0)\frac{\mathrm{d}x'(t')}{\mathrm{d}t'}\frac{\mathrm{d}t'}{\mathrm{d}H_c} \\
&= \left(\frac{H_c}{E_c(0)}\right)^{3/4}\frac{p'_x(0)p'_x(t')}{4}\frac{t}{E_c^{1/4}(0)H_c^{3/4}} = \frac{t\,p'_x(0)p'_x(t')}{4E_c(0)}.
\end{aligned}
\tag{38}
$$

Thus,

$$
\left\langle\frac{\partial}{\partial H_c}(p_x(0)x(t))\right\rangle_e = \frac{3\langle p_x(0)x(t)\rangle_e}{4E_c(0)} + \frac{t\langle p_x(0)p_x(t)\rangle_e}{4E_c(0)}
\tag{39}
$$

and, from (35) and (39),

$$
\phi_{xx}(t) = \frac{5}{4}\frac{\langle p_x(0)x(t)\rangle_e}{E_c(0)} + \frac{t\langle p_x(0)p_x(t)\rangle_e}{4E_c(0)}.
\tag{40}
$$

The simplification we obtained, due to the specific form of the Jacobian, is valid for a large class of Hamiltonians, including the billiards.

5. Short time dynamics

In order to derive some explicit formulas, we consider expressions (6) for the correlation functions derived in Section 2 by fitting the numerical results. For the NS we assume $\Omega \approx \omega$ and $\beta \approx \alpha$. Using Eqs. (19), (23) and (24) we obtain the following result:

$$
\langle E_o(t)\rangle = E_o(0) + \frac{\gamma^2}{m}(B + At + f(t) + g(t)),
\tag{41}
$$

where B is a constant, $f(t)$ is an oscillatory function and $g(t)$ is proportional to $e^{-\alpha t}$. The important result is the coefficient A

$$
A = 4\mu\omega\alpha\frac{[(\sigma/4\mu\omega)(\omega_0^2 + \omega^2 + \alpha^2) - E_o(0)/E_c(0)]}{[(\omega_0 - \omega)^2 + \alpha^2][(\omega_0 + \omega)^2 + \alpha^2]}.
\tag{42}
$$

For fixed oscillator frequency ω_0 and a given chaotic energy shell $E_c(0)$ (and, consequently, for given σ, μ, ω and α), the ratio $E_o(0)/E_c(0)$ is the responsible for the average increase or decrease of $\langle E_o(t)\rangle$. The short time equilibrium in the energy flow is given by the condition $A = 0$, or

$$
\frac{E_o(0)}{E_c(0)} = \frac{\sigma}{4\mu\omega}(\omega_0^2 + \omega^2 + \alpha^2).
\tag{43}
$$

We now turn to the equation of motion of $z(t)$ under the average effect of the interaction with the chaotic system. From Eqs. (15) and (22)

$$
\langle\ddot{z}(t)\rangle + \omega_0^2\langle z(t)\rangle = -\frac{\gamma}{m}\langle x(t)\rangle = \frac{\gamma^2}{m}\int_0^t \mathrm{d}s\,\phi_{xx}(t-s)\langle z(s)\rangle.
\tag{44}
$$

Integrating by parts yields

$$
\langle\ddot{z}(t)\rangle + \left(\omega_0^2 - \frac{\gamma^2 F(0)}{m}\right)\langle z(t)\rangle + \frac{\gamma^2}{m}\int_0^t \mathrm{d}s\,F(t-s)\langle\dot{z}(s)\rangle + \frac{\gamma^2}{m}z(0)F(t) = 0,
\tag{45}
$$

where

$$
F(t-s) = \frac{2\mu e^{-\alpha(t-s)}}{E_c(0)(\alpha^2 + \omega^2)}\{\omega\cos[\omega(t-s)] + \alpha\sin[\omega(t-s)]\}.
\tag{46}
$$

Eq. (45) shows that the interaction produces a harmonic correction to the original potential, a frequency-dependent dissipative term and an external force proportional to $z(0)$. The choice $z(0) = 0$ simplifies (45) and turns it into an average Langevin equation.

We note that the correction to the harmonic potential is repulsive and may qualitatively change the average motion of the oscillator if $\gamma^2 > m\omega_0^2/F(0)$. The critical perturbation $\gamma_c^2 = m\omega_0^2/F(0)$ is related to the stability of the equilibrium points of the full three-dimensional system, which becomes unstable for large γ, establishing a direction of unbounded motion. For the NS, for example, the origin $(x, y, z) = (0, 0, 0)$ becomes unstable at $\gamma^2 = 0.1 m/\omega_0^2$ (see Eq. (4)). The function $F(0)$, on the other hand, can be computed numerically from Eq. (46) and results in approximately 10. This value can also be estimated by noticing that $F(t)$ has dimensions of $\langle x(0)x(t)\rangle/E_c$, which, at $t = 0$, yields $2\langle x(0)^2\rangle/E_c = 1/0.1$. Therefore one must choose $\gamma < \gamma_c$ in order to consider the coupling as a perturbation to the original system, avoiding the introduction of these instabilities. We remark that similar corrections are observed in the theory of Caldeira and Leggett [9] and in the adiabatic calculations of Wilkinson–Berry–Robins [2].

Fig. 5 shows a comparison between the numerically calculated 'bare' oscillator energy $\langle E_o(t)\rangle$, where

$$E_o(z) = \frac{p_z^2}{2m} + \frac{m\omega_0^2 z^2}{2}, \tag{47}$$

the 're-normalized' oscillator energy $\langle E_{or}(t)\rangle$, where

$$E_{or}(z) = \frac{p_z^2}{2m} + \frac{m\omega_0^2 z^2}{2} - \frac{\gamma^2}{m}F(0)z^2, \tag{48}$$

and expression (41) without the oscillating term $f(t)$. We have chosen γ and m so that $\omega_0^2 - \gamma^2 F(0)/m > 0$. We also chose ω_0 so that $g(t)$ decreases very fast, i.e., $e^{-\alpha/\omega_0} \approx 10^{-4}$. In this case only the linear and the oscillating terms in Eq. (41) are important. We have subtracted the oscillating part of Eq. (41) in Fig. 5 to highlight the
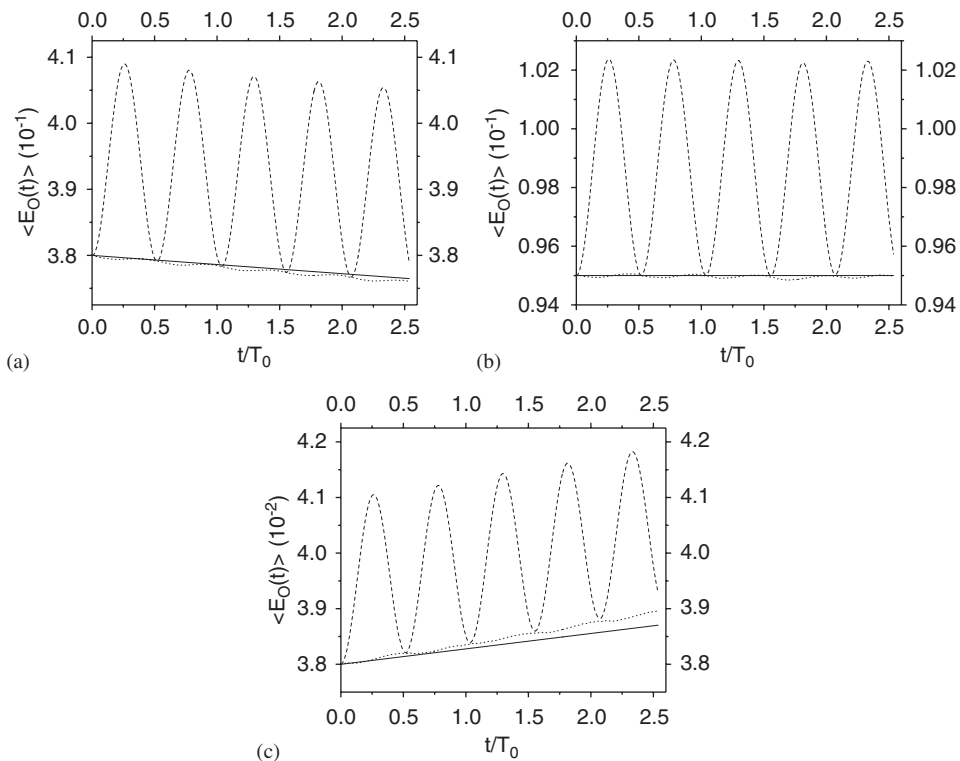

Fig. 5. Average oscillator energy at short times with the NS as chaotic system. The dashed line shows $\langle E_o(t)\rangle$ and the dotted line shows $\langle E_{or}(t)\rangle$, both obtained numerically. The full line shows Eq. (41) without $f(t)$. (a) $E_o(0)/E_c(0) = 1.0$, (b) $E_o(0)/E_c(0) = 0.25$ and (c) $E_o(0)/E_c(0) = 0.1$. The oscillator's parameters, coupling constant and number of initial conditions are the same as in Fig. 3.

linear increase or decrease in the average energy. In the time scale of Fig. 5, which corresponds to several periods of the decoupled oscillator, the linear behavior describes very well the numerical results. Fig. 5b shows the equilibrium situation according to Eq. (43).

For the QS the correlation functions can also be simplified with the assumption that $\beta \approx \alpha \approx \lambda$ (we shall keep for now $\omega \neq \Omega \neq v$). We get from (19), (23) and (24)

$$\langle E_o(t) \rangle = E_o(0) + \frac{\gamma^2}{m}(B' + A't + f'(t) + g'(t)), \tag{49}$$

where B' is a constant, $f'(t)$ is an oscillatory function and $g'(t)$ is a sum of terms proportionals to $\mathrm{e}^{-\alpha t}$ and $t\,\mathrm{e}^{-\alpha t}$. The coefficient A' is given by

$$A' = \frac{5\beta\Omega\alpha}{2} \frac{[(2\Lambda\sigma/5\beta\Omega)(\omega_0^2 + \omega^2 + \alpha^2) - \eta E_o(0)/E_c(0)]}{[(\omega_0 - \omega)^2 + \alpha^2][(\omega_0 + \omega)^2 + \alpha^2]}, \tag{50}$$

where

$$\Lambda = \frac{[(\omega_0 - \Omega)^2 + \alpha^2][(\omega_0 + \Omega)^2 + \alpha^2]}{[(\omega_0 - \omega)^2 + \alpha^2][(\omega_0 + \omega)^2 + \alpha^2]} \tag{51}$$

and

$$\eta = 1 + \frac{\mu[(\omega_0 - \Omega)^2 + \alpha^2][(\omega_0 + \Omega)^2 + \alpha^2]}{5\beta\Omega} \frac{[(\omega_0^2 + \kappa^2)^2 + 2v^2(\omega_0 + \kappa)(\omega_0 - \kappa) - 3v^4]}{[(\omega_0 - v)^2 + \kappa^2]^2[(\omega_0 + v)^2 + \kappa^2]^2}. \tag{52}$$

We note that if we set $\kappa = 0$ (which amounts to cancel the last term of (39)) and with $\omega = \Omega$, (50) becomes very similar to (42). For $\kappa \neq 0$, η can be positive or negative and if $\eta < 0$, $\langle E_o(t) \rangle$ cannot decrease with time. This possibility never happens for the NS. The equilibrium in the energy flow is given by the condition

$$\frac{E_o(0)}{E_c(0)} = \frac{2\Lambda\sigma}{5\beta\Omega} \frac{(\omega_0^2 + \omega^2 + \alpha^2)}{\eta}. \tag{53}$$

Eq. (45) remains valid for QS, but the function $F(t)$ is given by

$$F(t - s) = \int \mathrm{d}s\,\phi_{xx}(t - s)$$

$$= \frac{\mu(t - s)\mathrm{e}^{-k(t-s)}}{4E_c(0)(k^2 + v^2)}\{k\cos[v(t - s)] - v\sin[v(t - s)]\}$$
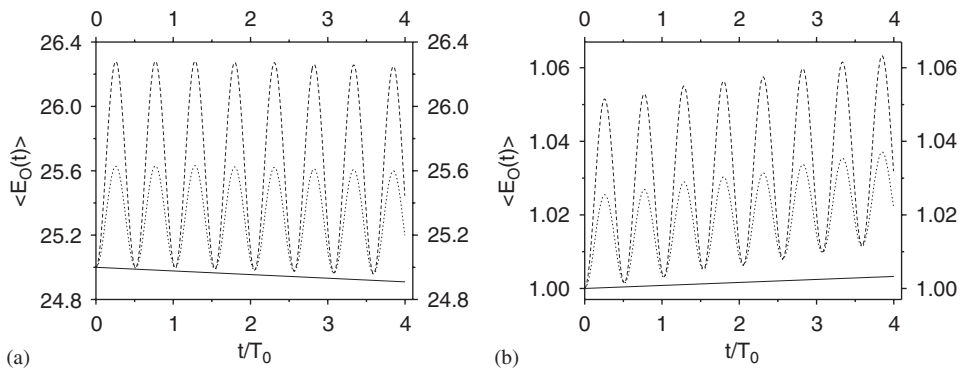
Fig. 6. Average oscillator energy at short times with the QS as chaotic system. The dashed line shows $\langle E_o(t) \rangle$ and the dotted line shows $\langle E_{or}(t) \rangle$, both obtained numerically. The full line shows Eq. (49) without $f'(t)$. (a) $E_o(0)/E_c(0) = 5.0$ and (b) $E_o(0)/E_c(0) = 0.2$. The oscillator's parameters, coupling constant and number of initial conditions are the same as in Fig. 4.
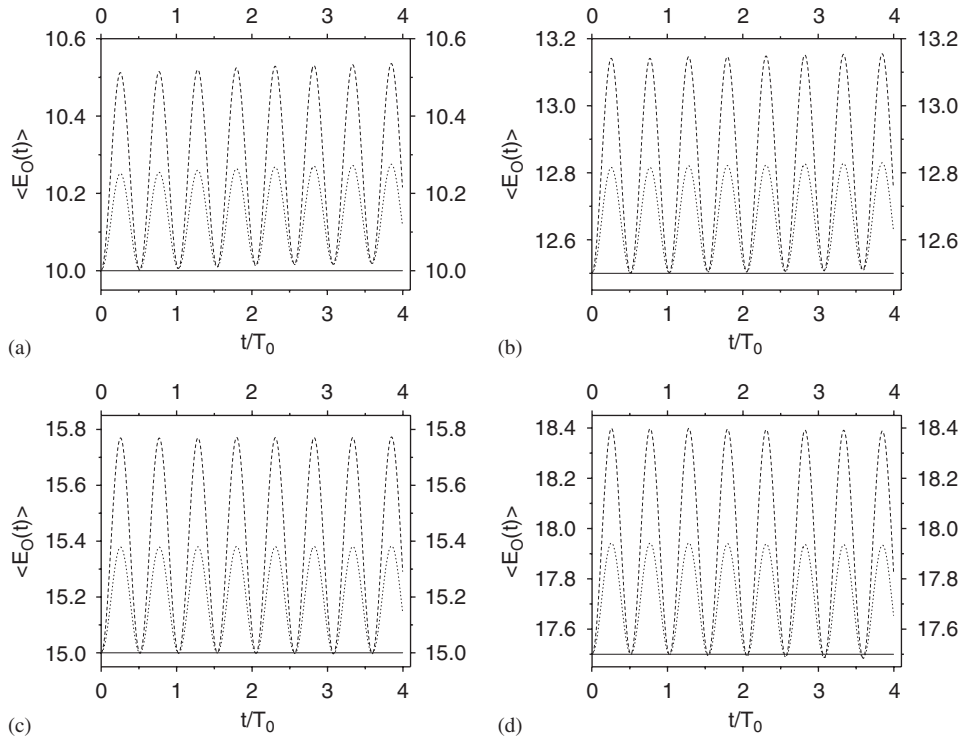
Fig. 7. Average oscillator energy at short times with the QS as chaotic system. The dashed line shows $\langle E_o(t)\rangle$ and the dotted line shows $\langle E_{or}(t)\rangle$, both obtained numerically. The full line corresponds to $E_o(0)$. (a) $E_o(0)/E_c(0) = 2.0$, (b) $E_o(0)/E_c(0) = 2.5$, (c) $E_o(0)/E_c(0) = 3.0$ and (d) $E_o(0)/E_c(0) = 3.5$. The oscillator's parameters, coupling constant and number of initial conditions are the same as in Fig. 4.

$$-\frac{\mu e^{-k(t-s)}}{4E_c(0)(k^2+v^2)^2}\{(v^2-k^2)\cos[v(t-s)]+2vk\sin[v(t-s)]\}$$

$$+\frac{5\beta e^{-k(t-s)}}{4E_c(0)(\Omega^2+k^2)}\{\Omega\cos[v(t-s)]+k\sin[v(t-s)]\}. \tag{54}$$

Fig. 6 shows the comparison between the numerically calculated $\langle E_o(t)\rangle$ and $\langle E_{or}(t)\rangle$ and expression (49) without $f'(t)$. The parameters were chosen in such a way that $e^{-\alpha/\omega_0} \approx 10^{-9}$ and $e^{-\alpha/\omega_0}/\omega_0 \approx 10^{-7}$, so that the linear and oscillating terms are the most important. Again we have subtracted $f'(t)$ to highlight the linear behavior and we have chosen γ and m so that $\omega_0^2 - \gamma^2 F(0)/m > 0$.

The agreement between numerical and analytical results for the equilibrium condition is only reasonable for the QS. The amplitudes of the oscillations in $\langle E_o(t)\rangle$ are bigger than in the NS case because of the different energy scale and coupling strength ($E_c(0) = 0.38$, $\gamma = 0.006$ for the NS and $E_c(0) = 5.0$, $\gamma = 0.01$ for the QS). In the approximation $\alpha = \beta = \lambda$ and $\omega \neq \Omega \neq v$, the equilibrium condition $A' = 0$ gives $E_o(0)/E_c(0) = 0.45$, but if we use $\omega = \Omega \neq v$, it changes to $E_o(0)/E_c(0) = 0.55$. This shows that small changes in the frequencies of the correlation functions change significantly the value of $E_o(0)/E_c(0)$ for the equilibrium condition. Fig. 7 displays $\langle E_o(t)\rangle$ for other values of $E_o(0)/E_c(0)$. It shows that the slope of the average energy as a function of the ratio $E_o(0)/E_c(0)$ has a very shallow minimum, which justifies its poor determination with linear perturbation theory. Fig. 7 suggests that the equilibrium condition may actually be located between $E_o(0)/E_c(0) = 2.5$ and $E_o(0)/E_c(0) = 3.0$.

6. Long time behavior and 'thermal' equilibrium

For long times the oscillator and the chaotic system reach an equilibrium, in the sense that their average energies tend to constant values. This is shown in Figs. 3 and 4 for the NS and the QS, respectively. Notice
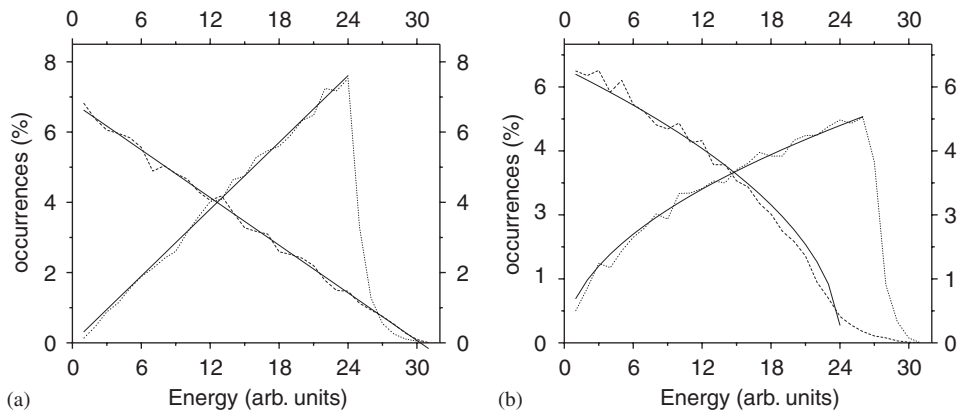
Fig. 8. (a) Distribution of energies of the ensemble of 30 000 trajectories at the time $t = 8 \times 10^5$, for the parameters of Fig. 3a. The full line shows a linear fitting, the dashed line shows E_o and dotted line shows E_c; (b) Distribution of energies at the time $t = 16 \times 10^5$, for the parameters of Fig. 4b and 30 000 trajectories. The full line corresponds to a square root fitting, the dashed line to E_o and the dotted line to E_c. The energy is shown in units of $E_T/30$.

that the energy ratio at the asymptotic equilibrium, at long times, is not necessary related to the equilibrium condition at short times, since non-linear effects are certainly important in the later. In order to characterize the equilibrium it is important to understand the energy distribution within each sub-system. It is particularly interesting to check if the oscillator follows Boltzmann's distribution.

To obtain the distributions numerically, we construct histograms in which the values of E_o and E_c are extracted from each trajectory of the ensemble for a fixed (long) time. The energy axis is divided into bins and the number of trajectories of the ensemble for which the oscillator's energy fall into each bin is counted. The same process is performed with respect to the chaotic system's energy. Fig. 8 shows the energy distributions thus obtained for the oscillator and the chaotic systems, NS and QS. The vertical axis shows the counts (occurrences) in percents of the total number of trajectories. In both cases it is clear that the oscillator does not follow the Boltzmann exponential law. This is actually not surprising, since the chaotic system is small (with only two degrees of freedom) and thus its energy is comparable to that of the oscillator. The Boltzmann distribution comes out naturally only when the system of interest is in contact with a reservoir of much larger energy (spread among its many degrees of freedom) [22], a condition that is not fulfilled here. In this context, it is also not clear whether a temperature can be defined in the present situation. We shall return to this question in a moment.

The energy distribution of the sub-systems can be totally understood in terms of their density of states. To see this, we assume that two hypothesis that are usual in the statistical physics of large systems [22] are also valid for our small systems. The first is to consider that, at equilibrium, all states of the full system, Eq. (1), are equally probable, i.e., the hypothesis of equal probabilities a priori. The second assumption concerns weak interactions between the sub-systems. The number of states $dN(E)$ *of the full system*, for which the oscillator has energy between E and $E + dE$, can be written in terms of the density of states

$$n(\varepsilon) = \int dV\, \delta(H(x, y, z) - \varepsilon), \tag{55}$$

where $H(x, y, z)$ is given by Eq. (1) and $dV = dx\, dp_x\, dy\, dp_y\, dz\, dp_z$. Neglecting the interaction potential V_I in Eq. (1), we can write $dN(E) = n_o(E)n_c(E_T - E)\, dE$, where n_o and n_c are the oscillator and chaotic system densities of states, respectively, and E_T is the total energy.

Based on these two assumptions, we can calculate the probability that the oscillator has energy between E and $E + dE$. The probability density $p_o(E)$ is, because of the first assumption, proportional to the number of states $dN(E)$:

$$p_o(E)\, dE \propto dN(E) = n_o(E)n_c(E_T - E)\, dE. \tag{56}$$

Likewise, the probability density of the chaotic system is

$$p_c(E)\, dE \propto n_c(E) n_o(E_T - E)\, dE. \tag{57}$$

We find that $n_o(E)$ is a constant and that $n_c(E) \propto E$ for NS and $n_c(E) \propto E^{1/2}$ for QS. Thus we obtain

$$\begin{cases} p_o(E) \propto (E_T - E) \\ p_c(E) \propto E \end{cases} \quad \text{for the NS} \tag{58}$$

and

$$\begin{cases} p_o(E) \propto (E_T - E)^{1/2} \\ p_c(E) \propto E^{1/2} \end{cases} \quad \text{for the QS.} \tag{59}$$

Of course these expressions are meaningful only for $0 < E < E_T$.

The full lines in Fig. 8 show a linear fitting for the case of the NS and a square root fitting for the QS. The fittings agree very well with the numerical results. The sudden decrease of the numerical distributions of the chaotic systems for high energies is due to the constraint that E_T is fixed.

Finally, with Eqs. (58) and (59), we can also calculate the oscillator's average energy in the equilibrium and compare the results with the values in Figs. 3 and 4. We have

$$\bar{E}_o = \int_0^{E_T} dE \, \frac{p_o(E)}{Z} \, E, \tag{60}$$

where the normalization constant

$$Z = \int_0^{E_T} dE \, p_o(E) \tag{61}$$

is $Z = E_T^2/2$ for the NS and $Z = 2E_T^{3/2}/3$ for the QS. We obtain $\bar{E}_o = E_T/3$ for NS and $\bar{E}_o = 2E_T/5$ for QS. From the probability densities p_c for NS and QS, we obtain $\bar{E}_c = 2E_T/3$ and $\bar{E}_c = 3E_T/5$, respectively, for NS and QS. For the parameters of Fig. 3a, $E_c(0) = 0.38$ and $E_o(0)/E_c(0) = 1.0$, we find $\bar{E}_o = 0.253$ and for Fig. 3b, where $E_c(0) = 0.38$ and $E_o(0)/E_c(0) = 0.1$, we get $\bar{E}_o = 0.139$. For Figs. 4a and b we obtain $\bar{E}_o = 12$ and $\bar{E}_o = 2.4$, respectively. In all cases the numerical values of the oscillator equilibrium energies are very close to the statistical prediction.

We now return to the question about the temperature. We will show that not only it is possible to define a temperature for our sub-systems but also that the equilibrium predicted by equating these temperatures results in the same partition of average energies predicted by the statistical analysis above. We first consider the usual definition of temperature given by

$$\frac{1}{T} = \frac{\partial S}{\partial E}, \tag{62}$$

where

$$S = k_B \ln n(E), \tag{63}$$

is the entropy and $n(E)$ is the density of states given by (55). The thermal equilibrium between the oscillator and the chaotic system implies $\partial S_o/\partial E_o = \partial S_c/\partial E_c$. However, because $n_o(E)$ does not depend on E, this gives $T_o = T_c = \infty$, and the equilibrium condition becomes useless.

Recent studies [13,23] have proposed modifications in the calculation of the entropy that, although irrelevant for large systems, make significant differences for small systems. Ref. [23] suggests dynamical corrections to the Boltzmann principle (Eq. (63)). On the other hand, Ref. [13] argues that the entropy in Eqs. (62) and (63) should be replaced by

$$S_\Phi = k_B \ln \Phi(E), \tag{64}$$

where $n(E) = d\Phi(E)/dE$. It has been shown that Eqs. (63) and (64) lead to identical results in the thermodynamic limit, but not for small systems, where (64) is still able to describe well the results of numerical simulations.

For the present model we have $\Phi_o(E) \propto E$, $\Phi_{NS}(E) \propto E^2$ and $\Phi_{QS}(E) \propto E^{3/2}$. In this framework, the equilibrium condition can be obtained equating the temperatures of the oscillator and the chaotic system, computed with the modified entropy Eq. (64). For the NS we find

$$\frac{\partial \ln \Phi_o(E_o)}{\partial E_o} = \frac{\partial \ln \Phi_{NS}(E_c)}{\partial E_c} \rightarrow \frac{E_o}{E_c} = \frac{1}{2} \tag{65}$$

and for the QS

$$\frac{\partial \ln \Phi_o(E_o)}{\partial E_o} = \frac{\partial \ln \Phi_{QS}(E_c)}{\partial E_c} \rightarrow \frac{E_o}{E_c} = \frac{2}{3}. \tag{66}$$

These results are in complete agreement with those obtained via the probability densities. Considering that our theoretical equilibrium energies describe very well the numerical calculations and the agreement between these energies and the thermal equilibrium conditions, we can conclude that the temperature T^Φ is indeed a good parameter for characterizing the equilibrium.

As a last remark we note that the initial distribution function, which is microcanonical only in the chaotic degrees of freedom, is not expected to evolve to a fully microcanonical distribution over the entire system. This is because the dynamics of the full system is probably mixed, not ergodic. However, the agreement of the energy distributions at long times with the above calculation suggests that the dynamics is at least 'close' to ergodic, in the sense that typical trajectories explore a large fraction of the available energy shell.

7. Conclusions

Our numerical results show that the coupling of an oscillator to a low-dimensional chaotic system simulates very well some aspects of a Brownian particle in a harmonic potential in the presence of a thermal bath. In particular, the average energy of the harmonic oscillator shows irreversible behavior and tends to an equilibrium at long times. For short time scales the average dynamics of the oscillator can be studied by linear response theory, where irreversibility is seen to result from the temporal decay of the chaotic correlations. The average motion of the oscillator follows a Langevin type of equation whose frequency-dependent dissipation depends only the amplitude, frequency and decay rate of the chaotic correlations.

The initial energies of the sub-systems play an important role in the problem. $E_c(0)$ defines the dynamical regime of the chaotic system, and the ratio $E_o(0)/E_c(0)$ defines if the oscillator will absorb or dissipate energy to the chaotic system. This is very similar to the usual thermalization of a Brownian particle, with $E_c(0)$ playing the role of the average energy $k_B T$ of the thermal bath. An important difference, however, is that the average energy of the chaotic system is also affected by the coupling, which is a direct consequence of its small number of degrees of freedom.

For long times the average energy of both oscillator and chaotic system tend to equilibrate. The value of the average energy and the energy distribution at equilibrium can be calculated assuming equal probabilities of the available microscopic states of the full system and weak coupling, so that probabilities over the full system can be approximated by the product of the probabilities over each sub-system.

It is interesting to see that both the short time and the long time analysis allow the determination of initial energies $E_o(0)$ and $E_c(0)$ such that no exchange of energies occur in the average. For short times it is given by the condition $A = 0$, Eq. (43) for the NS system. For the parameters of Fig. 3 it gives $E_o(0)/E_c(0) \approx 0.25$. For long times, since $\bar{E}_o = E_T/3$, imposing $\bar{E}_o = E_o(0)$ and $E_T = E_o(0) + E_c(0)$, we find $E_o(0)/E_c(0) = 0.5$. The two estimates clearly disagree, which means that for intermediate times non-linear corrections to the linear theory become important and change the short time tendency of the average energy. In other words, the short time dynamics is completely determined by the properties of the isolated chaotic system, whereas the long time behavior is dictated by the statistical properties of the full system. The same reasoning applies to the QS. Fig. 9 shows $\langle E_o(t) \rangle$ for $E_c(0) = 0.38$ and $E_o(0) = E_c(0)/2 = 0.19$. This is the condition for no exchange of energies at long times, but corresponds to a situation where the oscillator should dissipate at short times. And that is exactly what happens: the long time behavior displayed in Fig. 9a shows that the oscillator's average energy is indeed approximately equal to its starting energy. However, for short times, Figs. 9a and b, it clearly dissipates
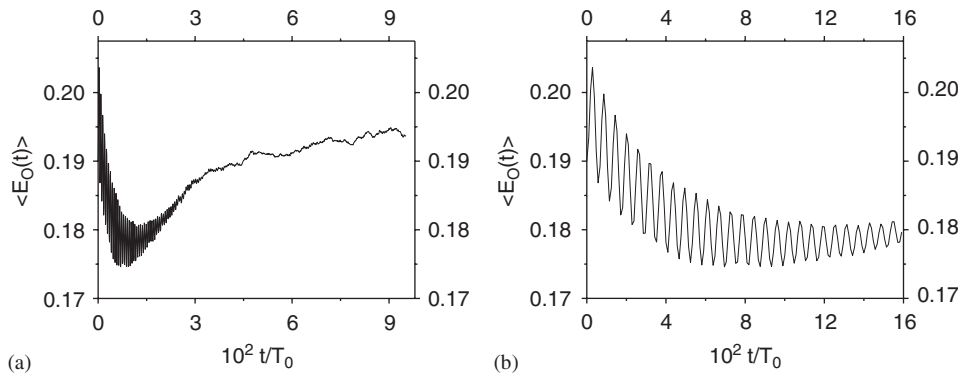
Fig. 9. (a) $\langle E_o(t) \rangle$ for $E_c(0) = 0.38$ and $E_o(0) = E_c(0)/2 = 0.19$. (b) Magnification of the time interval from 0 to 2×10^5 showing the initial dissipation of energy.

energy, re-absorbing it back latter on. The disagreement between short and long time equilibrium conditions may also be related to the low dimensionality of the chaotic system [21].

The low dimensionality is also reflected in the fact that the oscillator's equilibrium energy is different for each initial set up, clearly distinguishing the small chaotic environment from an infinite thermal bath. Besides, the oscillator's motion for a single realization (single initial condition) exhibits large fluctuations with respect to the average.

Finally, the temperature defined from the volume entropy S_Φ [13] describes very well the equilibrium condition of our model, since it agrees with the equilibrium conditions found by the probability densities and, consequently, with the numerical results. We believe this result contributes to the question of '…whether the volume or the area entropy is the correct starting point of thermostatistics for small systems' [13]. However, it is important to remember that, although it is possible to define such a temperature, there are always large fluctuations around these equilibrium values because of the low number of degrees of freedom. Moreover, it is not clear why the probability densities are given by the density of states and the temperature by its integral.

We note that the validity of the two assumptions we used here to describe the long time equilibrium regime are the subject of a debate that touches on the foundations of statistical mechanics [24]. The assumption of equal probabilities a priori is related to ergodicity and mixing. Here we have applied this condition without much justification, especially because the global system Eq. (1) is probably not mixing. However, the results we obtained from this assumption agrees very well with the numerical calculations. Thus, we believe that this type of low-dimensional models can be good testing grounds to the study of these topics.

Acknowledgements

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