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Eigenfunctions of the Liouville operator, periodic orbits and the principle of uniformity

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Abstract. We investigate the eigenvalue problem for the dynamical variables' evolution equation in classical mechanics,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \mathcal{L}f$$

where $\mathcal{L}$ is the Liouville operator, the generator of the unitary one-parameter group $U_t = \mathrm{e}^{-\mathcal{L}t}$. We show that the non-constant eigenfunctions are distributions on the energy shell and non-vanishing on its elementary retracing invariant submanifolds: rational tori for the integrable case or periodic orbits for the chaotic case.

The formalism unveils an equivalent statement, concerning the definition of a measure on the Hilbert space of dynamical variables, for the principle of uniformity. Introducing this measure, which is delta concentrated on the periodic orbits, we are able to derive the classical sum rules obtained from the principle of uniformity from the way the periodic orbits proliferate for increasing periods.

Introduction

Quantum mechanics has introduced, in physics jargon, the notion of Hilbert spaces; Hermitean operators have been named *observables* and several other concepts previously used mainly by mathematicians are now part of common usage in physics. Awareness of chaotic dynamics has further enlarged the language of physics with its manifolds, foliations, tori, and so on.

In this paper, the notion of Hilbert space within classical mechanics [2] is explored in detail for (bounded) autonomous Hamiltonian systems, with one and two degrees of freedom.

The original motivation of this paper was to put on a more solid basis the discussion in [3] concerning the linearization of Hamilton's equations, no matter how complex. The linear equations obtained in this formalism are the evolution equations for the eigenfunctions of the Liouville operator, which forms an infinite set.

In [3], we have considered the coordinate–coordinate coupling of a chaotic system to a 'system of interest'. Linearization of the chaotic part of Hamilton's equations is shown to be equivalent to the realization of a bath of oscillators representing the chaotic dynamics. The *spectral density* of the chaotic system at a specific energy is the relevant information for the construction of the bath (see section 2 later).

We can study dissipation of energy of the system of interest in this way, provided the coupling is weak. We therefore establish a connection between the Caldeira–Leggett

formalism [6], which models the environment as a bath of oscillators, and the one considered first by Wilkinson [4] and afterwards more rigorously by Berry and Robbins [5], which treats the environment as a microcanonical distribution of chaotic systems.

The eigenfunctions of the Liouville operator have an interest in themselves too, as we are going to show in the following, particularly because they are attached to the periodic orbits of the system. The periodic orbits of chaotic systems are largely used nowadays as elements to study the semiclassical limit of the corresponding quantum problem, its energy levels and much more. Here we are interested in the applications of periodic orbit theory in their natural habitat, classical mechanics. Curiously, the formalism we have to use to accomplish such a programme, Hilbert spaces, has quantum mechanics as a traditional partner.

We obtain a relation between the spectrum of the Liouville operator and periodic motion. One of our major conclusions is an alternative formulation of the principle of uniformity [1]. We derive a sum rule for the correlation function of classical variables in terms of periodic orbits. To accomplish this, a Dirac delta measure concentrated on the periodic orbits is defined [7], and their weights are determined by a feedback of the principle of uniformity.

This paper is organized as follows. In section 1, we explain the hypotheses which situate the eigenvalue problem. The following section is devoted to a careful analysis of the eigenfunctions of the one-dimensional problem, which, though simple, works as a guide to issues more relevant in higher dimensionalities, such as the linearization of Hamilton's equations. The two-dimensional problem is considered in section 3 for the ergodic and integrable cases. We also discuss its generalization to the mixed case. In section 4, the measure of the Hilbert space is presented in connection with the principle of uniformity. In section 5, we present the inner product. This is also where we obtain a sum rule for a classical correlation function in terms of periodic orbits.

The paper is not mathematically organized, in that the tools (measure and inner product) are defined after the natural physics reasoning suggests modification of the usual ones. To a mathematician, it may sound confusing to evoke a Hilbert space without specification of the inner product, and in their interest a more rigourous exposition of the mathematical ideas involved has been given elsewhere [7]. There are several contributions ([16] and references 3, 8, 10 therein, [17, 18]) which treat mathematical aspects of the spectral theory of dynamical systems. In particular, we can infer that the Liouville operator considered here is different from the usual one with Lebesgue measure in phase space, because they are not *unitary equivalent* [2]. We restate some of our conclusions in the final section.

1. Statement of the problem

We consider autonomous Hamiltonian systems

$$H = \frac{p^2}{2m} + V(q)$$

where $V(q)$ tends to infinity as $|q| \to \infty$ and has a lower bound. This guarantees that the solutions of the Hamilton equations are bounded in phase space. A more delicate implication is that, since $V(q)$ has a global minimum, Hamilton's equations have solutions that can be extended to infinity in time [9] for any initial condition, i.e. global solutions (for any initial condition, henceforth implicit). A potential for which the Hamiltonian has global solutions is called *complete*. There are hypotheses less restrictive where the potential completeness can be proven, but we shall not consider them here.

The first step in order to introduce the Hilbert space in classical mechanics is to move our attention to complex-valued functions of the phase space variables, the dynamical variables, rather than considering only the trajectories of the Hamiltonian flow. We start with the C^∞ functions of the phase space variables:

$$f : \mathbb{R}^{2N} \to \mathbb{C} \qquad (q, p) \to f(q, p).$$

In fact, we may deal with functions of compact support†, C_0^∞, and since we have assumed that the energy surfaces $E = H$ enclose a compact set of the phase space, there is no loss of generality. Their time evolution is given by the following relation, which defines the evolution operator U_t,

$$U_t(f(q, p)) = f(q(t), p(t))$$

where $(q(t), p(t))$ is the point of the unique trajectory with (q, p) as initial condition.

An equivalent way of obtaining the evolution of a dynamical variable is given by the differential equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \mathcal{L}f \tag{1}$$

where $\mathcal{L}$ is the Liouville operator defined by the classical Poisson brackets:

$$\mathcal{L}f = [f, H] = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}.$$

A sufficient condition for U_t being an *unitary one-parameter group* is that Hamilton's equations have global solutions, i.e. that the potential $V(q)$ be complete. The statement that U_t is unitary, known as the Liouville theorem, is given in the following [2].

Theorem. If Hamilton's equations possess global solutions, then $-\mathrm{i}\mathcal{L}$ is an essentially self-adjoint operator.

An *essentially* self-adjoint operator has an extension to the closure of its domain in which it is self-adjoint.

As exposed in the following, the non-trivial, i.e. non-constant, eigenfunctions of $-\mathrm{i}\mathcal{L}$ forces the introduction of a different inner product. We leave the details of its definition to the course of our exposition. It is clear that the proof of the above theorem [2] is dependent on the measure and the consequent inner product (self-adjointness). We have already shown that it remains valid with the Dirac measure to be defined [7].

The outline of the mathematical reasoning [7] is somewhat the reverse of the way these issues were revealed to us. Firstly, one has to define an *adequate* measure $\mathrm{d}\mu$ in the linear vector space of dynamical variables, then use the $L^2(\mathbb{R}^{2N}, \mathrm{d}\mu)$ inner product between the dynamical variables to obtain a Hilbert space. In this framework, one has to prove the quoted theorem.

Assuming we already have a Hilbert space, we can conclude that there is an infinite orthonormal basis in which the natural Hermitean operator in classical mechanics, $-\mathrm{i}\mathcal{L}$, is diagonal. This point has already been used to linearize Hamilton's equations of a chaotic flow[3]. Therefore, we shall study equation (1) as an eigenvalue problem.

† The support of a function f is the domain set where it is different from zero.

2. One-dimensional eigenfunctions

The integrability of one-dimensional autonomous Hamiltonian systems ensures the existence of a pair of action-angle variables, (J, θ), in terms of which the Hamiltonian can be written as

$$H = H(J).$$

Energy conservation specifies an action J once we start a trajectory in a phase space point (q, p). All trajectories are closed and periodic for Hamiltonians of the kind we are considering. The eigenvalue equation for the Liouville operator is written in action-angle variables as

$$-\mathrm{i}\mathcal{L}f = \lambda f \qquad \frac{\partial H}{\partial J}\frac{\partial f}{\partial \theta} = \mathrm{i}\lambda f \tag{2}$$

since H is independent of θ.

Writing $f(J, \theta) = g(J)t(\theta)$, the f dependence on J turns out to be a normalizing constant. Let the frequency of the movement for the energy shell $E = H(J)$ be $\omega(J) \equiv \partial H/\partial J$. The eigenfunctions of the Liouville operator are therefore given by

$$C(J)\exp\left(\mathrm{i}\frac{\lambda}{\omega(J)}\theta\right).$$

We must notice, however, that, since $f(\theta + 2\pi) = f(\theta)$, λ must be equal to an integer times the frequency $\omega(J)$,

$$\lambda \equiv n\omega(J) \tag{3}$$

a fact that restricts the eigenfunction to the energy shell. This property characterizes the eigenfunctions as distributions on the phase space: each eigenfunction resides in an energy curve, a set of measure zero, vanishing otherwise. We see, in practice, here what the closure of the C_0^∞ set has appended. The set of distributions is the closure of C_0^∞.

For each energy we have, therefore, an enumerable set of basis functions

$$f_{nJ_0}(\theta) = \mathrm{e}^{\mathrm{i}n\theta}\chi(J - J_0) \tag{4}$$

which is exactly the Fourier expansion basis and χ is the characteristic function of the energy curve. One can easily guess that the expansion of an arbitrary function $F(J, \theta)$ in terms of this basis is

$$F(J, \theta) = \sum_n C_n(J)\mathrm{e}^{\mathrm{i}n\theta}$$

where the coefficients are given by the relation

$$C_n(J) = \frac{1}{2\pi}\int_0^{2\pi} F(J, \theta)\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta.$$

But $C_n(J)$ must equal the projection of F onto an eigenfunction and the easiness of the formula above lies in the fact that the energy curve and the curve of constant J are equal sets. The coincidence of the sets $E =$ constant, $J =$ constant, and the closed orbit of period $2\pi/\omega(J)$ happens only in one dimension and in this there lurks an important property of the inner product. Since it has to cope with the distribution nature of the eigenfunctions a delta weight is necessary, otherwise any product of f_n with F would have zero integral. Therefore, a more elaborate formal calculation of C_n might be

$$C_n(J') = (F, f_{nJ'}) = \frac{1}{\Sigma}\int \delta(E - H)\bar{f}_{nJ'}(\theta)F(J, \theta)\mathrm{d}\theta\mathrm{d}J \tag{5}$$

where Σ is a normalization constant and $(\cdot,\cdot)$ is our notation for the inner product.

To calculate the time evolution of an arbitrary function F, we only have to use the unitary operator U_t.

2.1. Linearization of Hamilton's equations

If we define explicitly the time dependence of the theta variable, we obtain

$$F(J,\theta(t)) = \sum_n C_n(J)\mathrm{e}^{\mathrm{i}n(\theta_0+\omega(J)t)}.$$

Writing $C'_n(J,t) = C_n(J)\mathrm{e}^{\mathrm{i}n\omega(J)t}$, we may define

$$u_n = \mathrm{Re}\, C'_n \qquad v_n = \mathrm{Im}\ C'_n \tag{6}$$

obtaining

$$\begin{aligned} \dot{u}_n &= -n\omega(J)v_n \qquad & \dot{v}_n &= n\omega(J)u_n \\ \dot{u}_n &= -\lambda_n(J)v_n \qquad & \dot{v}_n &= \lambda_n(J)u_n \end{aligned} \tag{7}$$

where $\lambda_n(J) = n\omega(J)$. Since a function is completely determined by the coefficients of its series expansion, we just have to solve the system of equations (7) to determine the evolution of any dynamical variable F. We have, therefore, found an infinite set of *linear* equations whose solutions are directly related to those of the original nonlinear Hamilton equations.

Equations (7) can be given another interpretation, when one regards the coefficients as owners of the dynamics of a set of linear harmonic oscillators. This is how we can establish the connection between dissipation modelled either by coupling to a chaotic system [4, 5] or by coupling to a bath of oscillators [6]. Other implications of this connection are presented in [3].

Introducing the Hamiltonian

$$\mathcal{H}_J(\boldsymbol{u},\boldsymbol{v}) = \sum_n \frac{\lambda_n(J)}{2}(u_n^2 + v_n^2) \tag{8}$$

we see from the above reasoning that its infinite-dimensional flow has a bijection with the time evolution of any dynamical variable F of the original system, whose information is carried into $\mathcal{H}_J$ by the eigenvalues $\{\lambda_n(J)\}$ of the Liouville operator.

In short, given a dynamical variable F, with $F(t=0) = F_0(J,\theta_0)$, we decompose F_0 in Fourier series, obtaining a set of constants $(u_n(0), v_n(0))$, the initial conditions for the Hamiltonian $\mathcal{H}$, and the solution of the system (7) furnishes the time evolution of F.

A parallel with quantum mechanics is possible here. From the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$$

and the expansion of $|\psi\rangle$ in terms of the eigenfunctions of H

$$|\psi(t)\rangle = \sum_n C_n(t)|\phi_n\rangle$$

we see that $C_n(t)$ follows the equations of a set of linear harmonic oscillators,

$$\dot{C}_n = -\mathrm{i}(E_n/\hbar)C_n$$

so that a Hamiltonian such as (8) can be constructed [8].

Before studying the two-dimensional case, a glance over the trivial linear harmonic oscillator, $H = \omega J$, is advisable. The eigenvalue problem for the corresponding Liouville operator reads

$$\mathcal{L} f = \mathrm{i}\lambda f \qquad \omega \frac{\partial f}{\partial \theta} = \mathrm{i}\lambda f.$$

The adequate solution for this case forces the choice $\lambda = n\omega$, but since the frequency ω is the same for any energy, the corresponding eigenfunction, $N(n)\mathrm{e}^{\mathrm{i}n\theta}$, is valid for the whole phase space.

Another peculiarity appears when we write the eigenfunctions in terms of the canonical variables (q, p). Letting $z = p + \mathrm{i}m\omega q$, where m is the mass, we obtain a correspondence between the phase space and the complex plane. We verify that

$$f_n = N(n) z^n \qquad \text{and} \qquad f_{-n} = N'(n) z^{-n}$$

where N and N' are normalizing constants. We recognize the set $\{f_n, f_{-n}\}$ as the basis for the Laurent series expansions, which generate any C^∞ function.

3. Two-dimensional case

We now search for the eigenfunctions for the Liouville equation (equation (1)) for a general Hamiltonian with two degrees of freedom. We shall take a lesson from the one-dimensional case and think of the eigenfunctions as distribuitions on the phase space, non-vanishing only over the energy surface.

3.1. Integrable regime

We first consider the integrable case when H can be written in terms of two actions. Letting $H = H(J_1, J_2)$, the eigenvalue equation takes the form

$$\omega_1(J_1, J_2) \frac{\partial f}{\partial \theta_1} + \omega_2(J_1, J_2) \frac{\partial f}{\partial \theta_2} = \mathrm{i}\lambda f.$$

It can be easily solved by separation of variables, a method that gives for f

$$f(\theta_1, \theta_2) = C(J_1, J_2) \exp\left(\frac{\mathrm{i}\nu_1\theta_1}{\omega_1} + \frac{\mathrm{i}\nu_2\theta_2}{\omega_2}\right) \tag{9}$$

with $\nu_1 + \nu_2 = \lambda$. Again, the univaluedness of f implies that $\mathrm{i}\nu_1$ and $\mathrm{i}\nu_2$ are pure imaginary, $\nu_1 = n\omega_1$ and $\nu_2 = m\omega_2$.

These conditions give rise to an issue which did not appear in the one-dimensional case. Consider J_1, $E = H(J_1, J_2)$, n_o and m_o chosen so that

$$\lambda = n_o\omega_{1o} + m_o\omega_{2o}.$$

Varying J_1 in the same energy surface, and consequently J_2, we change to another torus, whose frequencies are ω_1 and ω_2. We must then find another pair of integers n, m satisfying

$$\lambda = n\omega_1 + m\omega_2$$

but this is not possible, except in pathologically simple cases, such as the two-dimensional linear harmonic oscillator. Departing from a torus where ω_{1o}/ω_{2o} is a rational number, the existence of another pair of integers n, m implies that ω_2/ω_{2o} and ω_1/ω_2 are also rational, conditions we cannot generally fulfil with only one parameter to vary.

Therefore, as the eigenvalue $\lambda = \lambda_{nm}$ is restricted to a torus, the associated eigenfunction is a distribution over this submanifold of the energy surface. We write the eigenfunctions in

terms of the characteristic functions of the tori, $\chi(J_1 - J_{1o})\chi(E - H(J_{1o}, J_2))$. The basis set for the two-dimensional integrable case is given by the two-dimensional Fourier basis times the characteristic functions

$$h_{J_{1o},nm}(J_1, J_2, \theta_1, \theta_2) = C(J_{1o}, J_{2o}) f_{nm}(\theta_1, \theta_2)\chi(J_1 - J_{1o})\chi(E - H) \tag{10}$$

where $f_{nm} = \mathrm{e}^{\mathrm{i}n\theta_1 + \mathrm{i}m\theta_2}$.

Up to now it seems that, within the energy surface $H = E$, we can move continuously along the tori, having an enumerable subset of the basis in each of them. This is not so however, because the eigenvalues for the irrational tori are all zero, and the corresponding *constant* eigenfunctions may be dropped from the basis. To show this, consider the time evolution of an eigenfunction on an irrational torus. Given $\theta_o = (\theta_{1o}, \theta_{2o})$ and $f_{nm}(\theta_o)$, we have

$$f_{nm}(\theta(t)) = f_{nm}(\theta_o)\exp(\mathrm{i}(n\omega_1 + m\omega_2)t).$$

We know that $\theta(t) \neq \theta_o$ for all t, but for

$$t_k = \frac{2k\pi}{n\omega_1 + m\omega_2}$$

the function f returns to its initial value, i.e. $f_{nm}(t_k) = f_{nm}(0)$. Thus we have a dense set of points on the irrational torus where the function f_{nm} assumes the same value. Being continuous on the torus, it must be constant and necessarily its eigenvalue is zero.

The point we shall consider next is how to expand an arbitrary function in terms of the basis formed by the Fourier-like functions on the rational tori alone. We need to show first that such an expansion is unique, once obtained.

Let G be a C_0^∞ function; define its restriction to the rational tori of a specific energy surface by

$$G_r(J_1, J_2, \theta_1, \theta_2) = \chi(E - H)\sum_k \chi(J_1 - J_{1k})G(J_1, J_2, \theta_1, \theta_2) \tag{11}$$

where $\{J_{1k}; k = 1, 2, \ldots\}$ is the list of rational tori. Evidently, each term of the above series has an unique expansion in terms of h_{mnk} (equation 10). Hence,

$$G_r(J_1, J_2, \theta_1, \theta_2) = \sum_{k,n,m} C_{nmk} h_{mnk}(J_1, J_2, \theta_1, \theta_2).$$

Since G_r is a faint, though dense, portrait of the original function G, a natural question is whether there can be another function F, with $F \neq G$, whose expansion is the same as G's. The answer is no because, once the expansions over the submanifolds are unique, $F_r = G_r$. This contradicts the fact that $F \neq G$, F and G continuous, implies $F \neq G$ in an open set which certainly intersects a rational torus.

The idea of such expansions is a bit different from the usual one, when we have a sequence of approximations, G_n, for a function G, defined on the whole domain. To obtain G in a point z' of an irrational torus, for example, we have to use the continuity of the function G,

$$\begin{aligned} G(z') &= \lim_{z \to z'} G(z) \\ &= \lim_{z \to z'} G_r(z) \end{aligned} \tag{12}$$

where in the last limit we take a sequence of $z \in$ rational torus.

The integrable case with separatrices in phase space, for example $H = H(J_1, J_2, \theta_1)$, presents no further difficulty, since the separatrices are, like the irrational tori, surrounded by rational tori, and we may use for them the same limiting process.

The coefficients of the expansions in terms of the two-dimensional basis are more difficult to obtain and it is not possible to guess them in a simple fashion as we did in the one-dimensional case. They depend on a definition of the inner product, which we shall examine in the last section.

3.2. Chaotic regime and periodic orbits

In the chaotic or mixed cases, no separation of variables is possible to solve equation (1). We shall use a bootstrap analysis (in the sense of partial differential equations) in order to find the eigenfunctions. Suppose we know a solution for the eigenvalue problem, ϕ, satisfying

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = [\phi, H] = \mathrm{i}\lambda\phi$$

where $H = H(J_1, J_2, \theta_1, \theta_2)$ is the Hamiltonian. The above equation has (generally) derivatives in three (or $2N - 1$, for N freedoms) independent directions contained in the energy surface, but we will examine it in the specific direction of a trajectory passing through the point $(J_{1o}, \theta_{1o}, J_{2o}, \theta_{2o}) = \boldsymbol{a}_0$.

We define a *projector*, Π,

$$\Pi(\boldsymbol{a}_0) : C^\infty(\mathbb{R}^{2N}) \to C^\infty(\mathbb{R})$$
$$\phi(\boldsymbol{J}, \theta) \to \phi(\boldsymbol{J}(s), \theta(s)) = g_{\boldsymbol{a}_0}(s) \tag{13}$$

which picks out a function along the orbit initiated at $\boldsymbol{a}_0$. In fact, Π is the same as the characteristic function of the trajectory through $\boldsymbol{a}_0$, but we shall keep calling it a projector, anticipating the analogy with quantum mechanics.

If $\boldsymbol{a}_0$ is a point of a periodic orbit of period T_a, we have $g_{\boldsymbol{a}_0}(s(T_a)) = g_{\boldsymbol{a}_0}(s(0)) = g_{\boldsymbol{a}_0}(0)$. Therefore, $g_{\boldsymbol{a}_0}$ may be expanded in a Fourier series with frequencies given by multiples of the frequency of the primitive periodic orbit, $2\pi/T_a$. Let us suppose that g has the following dependence with respect to $s = t$:

$$g_{\boldsymbol{a}_0}(s) = g_0 \exp \mathrm{i}\frac{2\pi n s}{T_a} \qquad \text{with } n \in \mathbb{Z}. \tag{14}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi(\boldsymbol{a}_0)\phi = \frac{\mathrm{d}}{\mathrm{d}t}g(s) \qquad \Pi(\boldsymbol{a}_0)\frac{\mathrm{d}}{\mathrm{d}t}\phi = \mathrm{i}\frac{2\pi n}{T_a}g(s) \qquad \lambda = \frac{2\pi n}{T_a} \tag{15}$$

and, therefore, the ϕ whose projections are given by equation (14) are eigenfunctions with eigenvalues given by equation (15).

The above equation proves the claim that the spectrum of the Liouville operator is related to the periods of closed orbits of the Hamiltonian flux.

We can also conclude that the eigenfunctions in the chaotic case are distributions over submanifolds of the energy surface. If we evaluate ϕ at a point $\boldsymbol{b}$ belonging to a periodic orbit of period $T_b \neq T_a$, we obtain

$$U_t(\phi(\boldsymbol{b})) = \phi(U_t(\boldsymbol{b})) \tag{16}$$

where U_t is the unitary operator $\exp -\mathcal{L}t$. Applying the above equation for $t = T_b$, the right-hand side results in $\phi(\boldsymbol{b})$ whereas the left-hand side furnishes, due to the fact that ϕ is an eigenfunction whose eigenvalue is $\lambda = 2\pi n/T_a$,

$$\exp\left(\frac{2\mathrm{i}\pi n T_b}{T_a}\right)\phi(\boldsymbol{b}).$$

Hence,

$$\left[\exp\left(\frac{2\mathrm{i}\pi n T_b}{T_a}\right) - 1\right]\phi(\boldsymbol{b}) = 0$$

and finally $\phi(\boldsymbol{b}) = 0$ for any $\boldsymbol{b}$ outside the trajectory of period T_a.

If a non-generic resonance occurs, $T_a/T_b = m/n$, ϕ could be non-vanishing at the periodic trajectory through $\boldsymbol{b}$. We can choose it to be vanishing, without any loss of generality, since this property would add no information to the basis of eigenfunctions. Indeed, the periodic orbit through $\boldsymbol{b}$ already has *its* Fourier-like basis; hence allowing some of the eigenfunctions ϕ to be non-vanishing in two (or finitely many) periodic orbits is clearly redundant.

Therefore, the eigenfunctions are non-vanishing exactly on the periodic orbits, and isolated in the chaotic and mixed cases.

Furthermore, if we try a non-vanishing eigenfunction on a point in the wandering set, we easily conclude, using equation (16), that it is a constant function in a dense subset of the energy surface, so that it is constant and its eigenvalue is zero. Therefore, these functions are superfluous for the basis in the same way that those residing in the irrational tori were.

3.3. Synthesis of the discussion

In the chaotic case, periodic orbits are isolated so that each eigenfunction is non-vanishing only on the orbit whose period is related to its eigenvalue via equation (15). In the other extreme, integrable Hamiltonians have one-parameter families of periodic orbits on the rational tori so that the eigenfunctions can be extended to them.

The eigenfunctions of the Liouville operator reside in periodic orbits, their eigenvalues are given by (15) for the chaotic, integrable and mixed cases. There is no reason whatever to exclude the mixed case, since the link between eigenfunctions and the Hamiltonian flux lies on the periodic orbits, which are present in all three cases.

As for the local family of each periodic orbit (rational torus), the energy is the parameter for a continuous set of eigenfunctions, which are family fixed. On the other hand, for a given energy, the set of eigenfunctions is enumerable. We will need this latter property to establish an order for the eigenfunctions, h_{nmk}, n, m as in equation (10) and $k = 1, 2, \ldots$ for decreasing eigenvalues, i.e. increasing periods.

Once we obtain coefficients for an expansion in terms of the eigenfunctions they are unique, because of the continuity of the expanded functions. How to determine these coefficients is a problem that depends on the way we evaluate inner products in this Hilbert space. The idea we will use to develop the inner product is based on the analogy with the projectors in quantum mechanics. What we need is a resolution of the identity like

$$\mathbf{1} = \sum_n |\varphi_n\rangle\langle\varphi_n|$$

where we have to substitute φ_n by the eigenfunctions of the Liouville operator. Notice, however, that the basis we obtained is enumerable for a specific energy surface, and, not least, the identity on the left-hand side is a faint portrait of this entire surface, surrounded by holes.

4. Connection with the principle of uniformity—the measure

The decomposition of the energy shell of an autonomous Hamiltonian system into submanifolds (periodic orbits) which densely cover it has already been explored in [1].

Hannay and Ozorio de Almeida proposed the *principle of uniformity*, which *grosso modo* states that the periodic orbits in a period interval $[T-\Delta T, T+\Delta T]$, though more numerous than the ones in an interval centred at a shorter period T', have a compensating smaller importance.

The content of this principle is used in the definition of the measure in the Hilbert space of dynamical variables. Before proceeding, let us mention some differences of approach. We will not need to smear delta functions around the periodic orbits and, consequently, use of the relations (12) and (13) of [1] is avoided. Instead, counting the periodic orbits is achieved directly in terms of delta functions of local variables, around each of them (or family of them, in a rational torus), constituting a different enunciation of the ideas developed in appendix 1 of [1]. Further, repetitions of periodic orbits are not counted, since the object with which we define the measure is the geometric trace of each orbit.

Let us first consider an example in one dimension for this kind of construction. We are going to obtain a delta weighted measure so that the integral of the characteristic function of the rationals in the interval [0, 1] amounts to one rather than zero, as the Lebesgue integral teaches us. Recall that the characteristic function of the rationals is given by

$$\chi_{\text{rationals}}(x) = \begin{cases} 1 & \text{if } x \in \boldsymbol{Q} \\ 0 & \text{otherwise.} \end{cases}$$

We seek a measure for which the equation

$$\int_0^1 \chi_{\text{rationals}}\, \mathrm{d}\mu = \int_0^1 \mathrm{d}x \tag{17}$$

holds. Of course, $\mathrm{d}\mu$ can be written in terms of delta functions on the rational numbers times the usual measure $\mathrm{d}x$:

$$\mathrm{d}\mu = \sum\nolimits''_{p,q} a_{pq}\delta(x-p/q)\mathrm{d}x$$

where $''$ is to remind us that p and q are coprimes. But we have to take care not to assign equal importance to all rational numbers, since this would make (17) diverge. A possible way to define $\mathrm{d}\mu$ is the following,

$$\begin{aligned} \mathrm{d}\mu &= k\sum\nolimits''_{p,q} \frac{1}{q^3}\delta(x-p/q)\mathrm{d}x \\ &= k\sum_q \frac{\varphi(q)}{q^3}\delta(x-p/q)\mathrm{d}x \end{aligned} \tag{18}$$

where k is a constant and φ is the Euler function which counts the number of coprimes of a given integer†. The above definition would render the integral (17) finite, so that we could choose k to make it equal one. However, we must point out that there are other choices for $\mathrm{d}\mu$ that produce a finite integral of the characteristic function of the rationals. Anyway, this simple example reveals that some hierarchy must be introduced in order to have convergence.

In our actual problem, the measure $\mathrm{d}\mu$ is determined by the Hamiltonian flow: chaotic, mixed or integrable. In the first two cases periodic orbits are isolated, whereas in the last case they appear in families over the energy shell. Correspondingly, the invariant manifolds have dimension 1, 1 or 2 (N for a $2N$-dimensional phase space), respectively.

† $\varphi(1) = 1$.

With the aid of the principle of uniformity, we establish the analogous relation that the measure for the energy shell has to fulfil. Let us denote by Σ the area of the energy surface

$$\Sigma(E) = \int \delta(E - H)\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{p}. \tag{19}$$

Our claim is that the principle of uniformity is equivalent to the following relation that defines the measure $\mathrm{d}\mu$,

$$\int \mathbb{1}\,\mathrm{d}\mu = \Sigma \tag{20}$$

where $\mathbb{1}$ denotes the characteristic function of the union of periodic orbits in a given energy surface. In words, equation (20) says that the integral of the characteristic function with the measure $\mathrm{d}\mu$ of the periodic orbits equals the area of the energy shell. To prove this claim we shall rederive the sum rule for the periodic orbits' intensities (cf equations (17) and (18) of [1]) from the way they proliferate with increasing period.

For the course of the present paper, we have to write down $\mathrm{d}\mu$ to normalize the eigenfunctions and then obtain the coefficients of the expansion of an arbitrary function.

4.1. Integrable case

For the integrable case, we have families of periodic orbits over the energy shell. The order in which we have numbered the eigenfunctions must be accompanied by the numbering of the tori, i.e. by decreasing frequencies. Infinite degeneracies, i.e. infinite tori where the respective periodic orbits have the same period, cannot be handled in this way. But these degeneracies will occur if, and only if, there is an *open* set of such tori, which finally means that the system is—within that open set—a two-dimensional linear harmonic oscillator. This exception will not concern us here, since it can be treated in a simpler fashion from the beginning (cf the one-dimensional example of a harmonic oscillator).

To select a family of orbits we introduce delta functions of the frequencies

$$\delta(\omega_1 - \omega_{1n})\delta(\omega_2 - \omega_{2n}).$$

So we have $\mathrm{d}\mu$ given by

$$\mathrm{d}\mu = \lim_{T\to\infty} \sum_{T_n<T} A_n \delta(\omega_1 - \omega_{1n})\delta(\omega_2 - \omega_{2n})\mathrm{d}^2 J\,\mathrm{d}^2\theta.$$

We obtain the weights of the tori from equation (20),

$$\begin{aligned}\Sigma &= \int \mathbb{1}\,\mathrm{d}\mu \\ &= \lim_{T\to\infty} \sum_{T_n<T} A_n \int \delta(\omega_1 - \omega_{1n})\delta(\omega_2 - \omega_{2n})\mathrm{d}^2 J\,\mathrm{d}^2\theta \\ &= \lim_{T\to\infty} \sum_{T_n<T} A_n 4\pi^2 \left|\det{}_n \left(\frac{\partial^2 H}{\partial J^2}\right)\right|^{-1}\end{aligned}$$

where the notation $\det_n$ means that the determinant is evaluated at the specific rational torus labelled n.

At this point, we make a choice of A_n which results in the sum rule for the periodic orbits, intensities. This choice is suggested by the results of Hannay and Ozorio de Almeida (see appendix 1 of [1]), even though there are other choices that verify (20). It should be

clear, however, that we are not departing from the principle of uniformity to derive it again, since our starting point is (20). We write

$$A_n = \frac{\Sigma}{K_{\rm int}(T)} \frac{{\rm d}T_n}{{\rm d}E} T_n^{-2} \tag{21}$$

where

$$K_{\rm int}(T) = \sum_{T_n<T} \left(\frac{{\rm d}T_n}{{\rm d}E}\right) \frac{(2\pi)^2}{T_n^2 |{\rm det}_n(\partial^2 H/\partial J^2)|}. \tag{22}$$

Assuming $K_{\rm int}(T) \sim T$ for large T, a fact we will prove in a moment, we have the following

$$\frac{\Sigma}{T} \sum_{T_n<T} \left(\frac{{\rm d}T_n}{{\rm d}E}\right) \frac{(2\pi)^2}{T_n^2 |{\rm det}_n(\partial^2 H/\partial J^2)|} \overset{T\to\infty}{\sim} \Sigma \tag{23}$$

which is the classical sum rule obtained from the principle of uniformity.

We obtain the asymptotic behaviour of $K_{\rm int}(T)$ using the fact that the density of periodic orbits grows with S^2, the square of the action, for large S [10]. Therefore

$$K_{\rm int}(T) \approx \int^T \left(\frac{{\rm d}T}{{\rm d}E}\right) \frac{(2\pi)^2}{T^2 |{\rm det}_T(\partial^2 H/\partial J^2)|} \rho(T){\rm d}T$$

where $\rho(T){\rm d}T$ is the number of periodic orbits between T and $T + {\rm d}T$. Denoting by $N(S) = aS^2$ the number of periodic orbits for large action, we obtain

$$\begin{aligned} \rho(T){\rm d}T &= 2aS \left|\frac{{\rm d}S}{{\rm d}E}\frac{{\rm d}E}{{\rm d}T}\right| {\rm d}T \\ &= 2aT^2(\omega_1 J_1 + \omega_2 J_2) \left|\frac{{\rm d}E}{{\rm d}T}\right| {\rm d}T. \end{aligned}$$

Substituting in the asymptotic expression for $K(T)$, we see that, since $(\omega_1 I_1 + \omega_2 I_2)$ and $|\,{\rm det}_T\,| > 0$ are smooth functions on the energy (compact) surface, their ratio may be taken outside the integral sign, using the intermediate value theorem for integrals, giving another multiplying constant. Therefore $K_i(T) \sim T$ for large periods.

Summing up, we rederived the asymptotic behaviour of the sum of periodic orbit intensities in the integrable case (cf equation (18) of [1]) from equation (20) and the proliferation of periodic orbits in the integrable regime.

We can now write the definition of the measure for the integrable case

$${\rm d}\mu = \lim_{T\to\infty} {\rm d}\mu_T \tag{24}$$

where

$${\rm d}\mu_T = \frac{\Sigma}{4\pi^2 K_{\rm int}(T)} \sum_{T_n<T} \frac{{\rm d}T_n}{{\rm d}E} T_n^{-2} \delta(\omega_1 - \omega_{1n})\delta(\omega_2 - \omega_{2n}){\rm d}^2 J {\rm d}^2\theta. \tag{25}$$

The above expression for ${\rm d}\mu_T$ has a ratio of divergences, therefore extra care is necessary when taking the limit $T \to \infty$: simultaneously on $K_{\rm int}(T)$ and the rest of the expression. We discuss this point further at the end of this section, where we give possible interpretations.

4.2. Chaotic case

For the chaotic case, periodic orbits appear isolated over the energy shell. We may enumerate the period of the orbits. Let Γ_n be a domain of phase space containing just one periodic orbit of period T_n. The union of these open sets is an open cover of the phase space. Notice that only primitive periodic orbits enter into the definitions of these sets, since we have to consider their geometrical location in phase space. For each Γ_n, we have a quadruple of local canonical variables: $H, T, {}_nq, {}_np$. We shall drop the index n of q and p hereafter. The first canonical pair is formed by H, the Hamiltonian function, and T, the time along orbits. The canonical pair, q, p, are the complementary coordinates, such that $(q - q_t)$ and $(p - p_t)$ measure the deviation of a point in phase space from the orbit.

To pick up the orbit of period T_n in the domain Γ_n, we need a three-dimensional delta function

$$\delta(H - H_t)\delta(q - q_t)\delta(p - p_t)$$

where (H_t, T_t, q_t, p_t) is the point of a trajectory beginning in an arbitrary point (with $t = 0$) inside Γ_n. The last two δ functions select the local family of periodic orbits in phase space, i.e. with varying energy. As previously said, Γ_n contains but one trajectory of period T_n, which is exactly the one selected by the first delta function.

Therefore, we can write $\mathrm{d}\mu$ as

$$\mathrm{d}\mu = \lim_{T\to\infty} \sum_{T_n<T} A_n \delta(H - H_t)\delta(q - q_t)\delta(p - p_t)\,\mathrm{d}H\,\mathrm{d}T\,\mathrm{d}q\,\mathrm{d}p.$$

The weights A_n are again obtained from equation (20),

$$\begin{aligned}\Sigma &= \sum_n A_n \int_{\Gamma_n} \delta(H - E)\delta(q - q_t)\delta(p - p_t)\,\mathrm{d}H\,\mathrm{d}T\,\mathrm{d}q\,\mathrm{d}p \\ &= \sum_n A_n T_n |\det{}_n(\mathbf{M} - \mathbf{I})|^{-1}\end{aligned}$$

where $\mathbf{M}$ is the reduced monodromy matrix, without the unit 2×2 block, and $\mathbf{I}$ is the unit matrix.

Following the same structure as the integrable case discussion, A_n is chosen so that it results in the sum rule for the periodic orbits' intensities. We emphasize again that this choice is suggested by the results of Hannay and Ozorio de Almeida (see appendix 1 of [1]), but this does not mean that we are departing from the principle of uniformity to deduce it.

$$A_n = \frac{\Sigma}{K_{\mathrm{cha}}(T)} \tag{26}$$

where

$$K_{\mathrm{cha}}(T) = \sum_{T_n<T} \frac{T_n}{|\det{}_n(\mathbf{M} - \mathbf{I})|} \tag{27}$$

and similarly the asymptotic behaviour $K_{\mathrm{cha}}(T) \sim T$ for large T gives the classical sum rule obtained from the principle of uniformity:

$$\frac{\Sigma}{T} \sum_{T_n<T} \left(\frac{T_n}{|\det{}_n(\mathbf{M} - \mathbf{I})|} \right) \overset{T\to\infty}{\sim} \Sigma. \tag{28}$$

For the chaotic case, one can find the proof for $K_{\mathrm{cha}}(T) \sim T$ in [11], based on the exponential proliferation of periodic orbits. Here we are emphasizing that this proliferation,

together with equation (20), *implies* the sum rules for chaotic and integrable cases, a relation we have not found elsewhere.

Hence, for a chaotic Hamiltonian, $\mathrm{d}\mu$ reads

$$\mathrm{d}\mu = \lim_{T\to\infty} \mathrm{d}\mu_T$$
$$\mathrm{d}\mu_T = \frac{\Sigma}{K_{\mathrm{cha}}(T)} \sum_{T_n<T} \delta(H-E)\delta(q-q_t)\delta(p-p_t)\,\mathrm{d}H\,\mathrm{d}T\,\mathrm{d}q\,\mathrm{d}p. \tag{29}$$

As in equation (25), we have in the chaotic case measured a ratio of divergences, which has to be computed before the limit of all orbits is taken.

Since periodic orbits appear isolated on the energy shell in the mixed case too, a brief digression on it is possible here. In general, bifurcations make the eigenvalues of $\mathbf{M}$ have modulus 1, which causes $\det(\mathbf{M}-\mathbf{I}) = 0$. The transition from integrability to ergodicity is characterized precisely by innumerous bifurcations of families of stable orbits. In fact, we have departed from an assumption about the domain Γ_n which is not true in the mixed case, because at the energy where the bifurcation occurs every open set contains the *two* arising families of periodic orbits. One has to take this into account in order to derive a measure for this case.

4.3. Discussion

To define the measure satisfying (20), we have some freedom to choose the weights of each periodic orbit. The particular choice made implied, using the way the periodic orbits proliferate with increasing periods, the principle of uniformity. This is how this section can be seen as a different deduction of uniformity, but evidently we could not have made that choice if we did not know the principle beforehand.

An alternative interpretation of the measure $\mathrm{d}\mu$ is possible. $\mathrm{d}\mu$ is given in terms of periodic orbits on a fixed energy surface, for both chaotic and integrable cases. In the equation (17) of [1], on the left-hand side,

$$\int \frac{\delta(\boldsymbol{r}_0-\boldsymbol{r}_t)}{\langle\delta(\boldsymbol{r}_0-\boldsymbol{r}_t)\rangle_t}\delta(H(\boldsymbol{r}_0)-E)\,\mathrm{d}\boldsymbol{r}_0^{2N} = \sum_j I_j\delta(T-T_j)$$

the first term of the integrand selects periodic orbits irrespective of their energy, and the other delta function fixes the energy shell. When averaged in time, the left-hand side adds up the contributions of all the periodic orbits of a fixed energy shell, which is precisely what the above-defined measure $\mathrm{d}\mu$ does.

As already defined explicitly, the sum rule follows

$$\Sigma = \int \mathrm{d}\mu = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sum_j I_j\delta(T-T_j) \tag{30}$$

i.e.

$$\sum_{|T_j|<T} I_j \xrightarrow{T\to\infty} 2T\Sigma. \tag{31}$$

A drawback that affects both treatments is the ratio of divergences: we have substituted a quotient of delta functions by one of divergent series. However, the formalism we have presented gives the sum rules directly in terms of primitive periodic orbits, and the disputable argument that 'almost all long orbits are primitive' is no longer necessary. This was possible because the measure $\mathrm{d}\mu$ is defined in terms of the geometric traces of the orbits, which are the primitive periodic orbits.

The idea that the sum rule (31) is asymptotic, independent of some short periodic orbits of a specific system, does not contradict the definition of the measure $\mathrm{d}\mu$. Indeed, one could let out of $\mathrm{d}\mu$ a finite number of periodic orbits (or rational tori) and one would continue to have a dense subset of the energy shell.

For practical purposes, however, one has to use an approximation $\mathrm{d}\mu_T$ for the whole dense measure. This could be somewhat formalized by noticing that a finite cover of neighbourhoods Γ_n is enough for a compact energy surface. Subject to this cover, one then defines a partition of unity [15] (the unity is the energy surface), which turns out to be very similar an idea to *'smearing around* (but finitely many) *periodic orbits'* [1].

On the other hand, the above reasoning leads to an apparent contradiction with the asymptotic character of the sum rule: the long periodic orbits are responsible for the generic behaviour of the sum of intensities, but $\mathrm{d}\mu_T$ takes into account a finite number of orbits.

Our explanation for this situation is that the short periodic orbits create something similar to grooves on the energy shell. Around their traces longer periodic orbits, with periods as large as desirable, accumulate. Thus, if one wishes to consider the *infinity* contribution, i.e. long periodic orbits, one can neglect a number of short periodic orbits. However, near each shortest orbit, there will always remain another shortest orbit of that region, which will be its heir. That is why we have mentioned a *hierarchy* of periodic orbits. In this way, though non-generic, these periodic orbits (the shortest in each region) may stand for the whole dense set of periodic orbits on the energy shell.

This analysis is widely known in the context of the Selberg trace formula and named *bootstrapping* [12]. Here we have a quantitative element to apply it, namely, the smaller the diameter d of the neighbourhoods Γ_n of an open cover, the more periodic orbits we need.

5. Internal product—applications

In this section, we study the consequences of the definition of the internal product based on the measure obtained for the chaotic case. Firstly we normalize the eigenfunctions to derive afterwards the correlation of a classical dynamical variable in terms of its value on the periodic orbits. We write $\mathrm{d}\mu$ as if $K_{\mathrm{cha}}(T)$ converges; the limits are implicit.

Given the measure $\mathrm{d}\mu$, the internal product is given by

$$\langle\cdot,\cdot\rangle : C_0^\infty(\mathbb{R}^{2N}) \times C_0^\infty(\mathbb{R}^{2N}) \to \mathbb{C} \tag{32}$$

$$(f_1, f_2) \to \langle f_1, f_2\rangle = \int f_1(\boldsymbol{q},\boldsymbol{p})\bar{f}_2(\boldsymbol{q},\boldsymbol{p})\mathrm{d}\mu. \tag{33}$$

The eigenfunctions are given by

$$f_{ni}(T) = N_i \exp\left(\frac{2\pi \mathrm{i} n T}{T_i}\right)\chi_i(\boldsymbol{q},\boldsymbol{p})$$

where χ_i is the characteristic function of the periodic orbit and N_i is a normalizing constant. We find the normalization factor from the following equation:

$$\|f_{ni}\|^2 = \langle f_{ni}, f_{ni}\rangle = 1.$$

We have restricted our attention to the chaotic case, but the integrable eigenfunctions can be treated in an analogous manner. N_i is given by

$$N_i = \left(\frac{K_{\mathrm{cha}}|\det(\mathbf{M}_i - \mathbf{I})|}{\Sigma T_{a_i}}\right)^{1/2}.$$

Notice that N_i is the same for all eigenfunctions based on the same orbit. Strictly speaking, the normalizing constant N diverges, but this divergence cause little harm as any manipulation with the eigenfunctions—expansions, norms and distances—has to be performed with the measure $\mathrm{d}\mu$, which cancels it out (see below). One might also overcome this divergence by using a finitely summed K_{cha}.

We can finally write the general eigenfunction for the Liouville operator of a chaotic Hamiltonian:

$$f_{ni}(T) = \left(\frac{K_{\mathrm{cha}}|\det(\mathbf{M}_i - \mathbf{I})|}{\Sigma T_i}\right)^{1/2} \exp\left(\frac{2\pi \mathrm{i} n T}{T_i}\right) \chi_i(\boldsymbol{q}, \boldsymbol{p}). \tag{34}$$

Now let $g(\boldsymbol{q}, \boldsymbol{p})$ be a dynamical variable. We can study its correlation in terms of the eigenfunctions of the Liouville operator. Let us write the expansion of g in terms of the eigenfunctions

$$g(\boldsymbol{q}, \boldsymbol{p}) = \sum_{n,i} C_{ni} f_{ni} \tag{35}$$

where C_{ni} is given by

$$C_{ni} = \int g(\boldsymbol{q}, \boldsymbol{p}) \bar{f}_{ni} \, \mathrm{d}\mu.$$

Firstly, we note that the term $f_{ni}\mathrm{d}\mu$ contains a product of distributions of the type $\chi\delta$, well defined because χ is finite. We obtain for this product

$$f_{ni}\,\mathrm{d}\mu = \left(\frac{\Sigma T_i}{K_{\mathrm{cha}}}|\det(\mathbf{M}_i - \mathbf{I})|\right)^{1/2} \frac{1}{T_i} \exp\left(-\frac{2\pi \mathrm{i} n T}{T_i}\right) \delta(H - E)\delta(q - q_t)$$
$$\times \delta(p - p_t)\,\mathrm{d}q\,\mathrm{d}p\,\mathrm{d}H\,\mathrm{d}T.$$

Writing $\tilde{g}_i(T) = g(\boldsymbol{q}(T), \boldsymbol{p}(T))$, the value the dynamical variable takes on the orbit labelled i, we obtain

$$C_{ni} = \left(\frac{\Sigma T_i}{K_{\mathrm{cha}}|\det(\mathbf{M}_i - \boldsymbol{I})|}\right)^{1/2} \frac{1}{T_i} \int_0^{T_i} \exp\left[-\frac{2\pi \mathrm{i} n T}{T_i}\right] \tilde{g}_i(T)\,\mathrm{d}T. \tag{36}$$

The correlation function

$$\mathcal{C}_g(t) = \frac{1}{\Sigma} \langle g(\boldsymbol{q}, \boldsymbol{p}), \bar{g}(\boldsymbol{q}(t), \boldsymbol{p}(t))\rangle \tag{37}$$

may be calculated in terms of the values of g on the periodic orbits only, with the measure $\mathrm{d}\mu$. The crucial point here is that, depending on whether g is the whole function (defined everywhere in phase space) or its projection onto the Liouville eigenfunctions basis, we have to write the internal product on (37) with the adequate measure. Of course, if we evolve the dynamical variable g, $C_g(t)$ may be obtained by the microcanonical average

$$\frac{1}{\Sigma} \int g(\boldsymbol{q}, \boldsymbol{p}) \bar{g}(\boldsymbol{q}(t), \boldsymbol{p}(t)) \delta(E - H)\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{p}$$

and this has the same value if we project g on the union of periodic orbits. In so doing, we need to evaluate

$$\frac{1}{\Sigma} \int g(\boldsymbol{q}, \boldsymbol{p}) \bar{g}(\boldsymbol{q}(t), \boldsymbol{p}(t))\,\mathrm{d}\mu.$$

As we have done in the one-dimensional study, we redefine the coefficients of the expansion of g so that they include the time dependence:

$$g(\boldsymbol{q}(t), \boldsymbol{p}(t)) = \sum_{n,i} C'_{ni}(t) f_{ni}(0).$$

Hence

$$\mathcal{C}_g(t) = \frac{1}{\Sigma}\sum_{m,j}\sum_{n,i} C_{ni}(0)\bar{C}'_{mj}(t)\langle f_{ni}, f_{mj}\rangle$$
$$= \sum_{n,i} \frac{T_i}{K_{\mathrm{cha}}|\det(\mathbf{M}_i - \mathbf{I})|} \exp\left[\frac{2\pi \mathrm{i} n t}{T_i}\right] |G_{ni}|^2 \tag{38}$$

where G_{ni} is the nth Fourier component of g along the orbit i:

$$G_{ni} = \frac{1}{T_i}\int_0^{T_i} \exp\left[-\frac{2\pi - \mathrm{i} n T}{T_i}\right] \tilde{g}_i(T)\,\mathrm{d}T. \tag{39}$$

The correlation (38) is in fact real because $\mathcal{C}_g(t) = \mathcal{C}_g(-t)$. This result can be applied or, rather, tested for a chaotic Hamiltonian whose periodic orbits are known. We can expect that the series for $\mathcal{C}_g$ is rapidly convergent for chaotic systems because of their rapidly increasing instabilities. We can also evaluate the Fourier transform of $\mathcal{C}_g$ obtaining that the spectral density,

$$S_g(\omega) \equiv 4\,\mathrm{Re}\int_0^\infty \mathrm{e}^{-\mathrm{i}\omega t}\mathcal{C}_g(t)\ \ \mathrm{d}t$$

is given by

$$S_g(\omega) = 4\pi \sum_{ni} \frac{|G_{ni}|^2 T_i}{K_{\mathrm{cha}}|\det(\mathbf{M}_i - \mathbf{I})|}\delta\left(\omega - \frac{2\pi n}{T_i}\right) \tag{40}$$

which has already appeared elsewhere [3].

5.1. Comment on the weights of periodic orbits

To define the measure $\mathrm{d}\mu$ both for chaotic and integrable cases, a choice of weights of periodic orbits was necessary. There are many other choices that result in (20), for example dividing A_n by T_n^2 to make K_{cha} (K_{int}) converge. This is an inconvenient freedom, because we could deduce one different classical sum rule from each choice.

The expansions of the functions, (35), are unchanged, because the coefficients cancel the normalizing constant. Those Fourier-like series are therefore invariant. On the other hand, the formula for the correlation of a dynamical variable, (38), is not invariant if we change the weights of the periodic orbits. Explicitly, if we define weights A'_n by

$$A'_i = \alpha_i A_i$$

then the correlation $\mathcal{C}_g(t)$ is changed to

$$\mathcal{C}'_g(t) = \sum_{n,i} \frac{\alpha_i T_i}{K'_{\mathrm{cha}}|\det(\boldsymbol{M}_i - \boldsymbol{I})|} \exp\left[\frac{2\pi \mathrm{i} n t}{T_i}\right] |G_{ni}|^2 \tag{41}$$

where K'_{cha} is given analogously by

$$K'_{\mathrm{cha}}(T) = \sum_{T_i < T} \frac{\alpha_i T_i}{|\det(\mathbf{M}_i - \mathbf{I})|}$$

(of course we are not considering all the α_i's to be equal to a constant).

A usual trick in semiclassical mechanics (Gutzwiller's trace formula [10]) can also be applied here to make K_{cha} converge, by setting $\alpha_i = \mathrm{e}^{-\epsilon T_i}$. In this case, to take the limit $\epsilon \to 0$ after using the series (38) is clearly equivalent to considering a finite number of orbits.

It is also easy to see that (α_i) viewed as a sequence of real positive numbers has only three relevant possible behaviours as $i \to \infty$. If it is bounded, it suffices to consider a sequence converging to unity or zero. From this we see that the notion of *hierarchy* of periodic orbits makes sense mathematically: the longer orbits can never surmount individually the weight of the finite number of orbits before them. Only if (α_i) is not bounded is this hierarchy modified. As in the case of α_i going to zero, the asymptotic behaviour of K_{cha} is changed when $\lim_{i\to\infty} \alpha_i = \infty$. It is an open question whether all possible choices for α_i make sense physically; numerical evidence, via equation (38), is being searched for to clarify this point.

For $\alpha_i = 1$ and setting the dynamical variable equal to a constant reduces formula (38) to the known classical sum rule of periodic orbit intensities [1], but for a general dynamical variable it provides a method of calculation of its correlation, apart from the usual one from the microcanonical ensemble.

6. Concluding remarks

We have thoroughly studied the eigenvalue problem for the Liouville operator in one and two dimensions. To our knowledge, a brief note about this subject has appeared in [13, 14]. There, only continuous eigenfunctions are considered, and consequently they are constant everywhere. We have allowed the eigenfunctions to have a distribution nature—proportional to characteristic functions of the invariant submanifolds of the energy surface—thereby obtaining non-trivial ones.

The applications we have developed for this approach belong to the domain of classical mechanics, particularly to the problem of coupling a chaotic system to a mesoscopic one [3]. A test of equation (38), a correlation of a classical dynamical variable, is in progress.

We have also shown that the principle of uniformity follows from a particular choice of Dirac measure over the periodic orbits [7] and no hypothesis about interchangeable limits, large periods and degree of smearing of delta functions was necessary (see in [1]). This permitted some steps towards different interpretations and applications. For example, an auxiliary quantity for the application of the sum rules was introduced in section 4.3: the diameter of the neighbourhoods of the periodic orbits. As the factors of the classical sum rules are similar to the ones in the Gutzwiller trace formula, the bootstrapping analysis might apply to both. The accuracy of the semiclassical analysis based on a number of periodic orbits would therefore be related to the farthest distance between them, i.e. the largest diameter of the neighbourhoods Γ_n.

Moreover, a sum over repetitions of a few periodic orbits, not sufficiently spread on the energy surface, would give better results for the eigenvalues whose eigenfunctions are localized around them. This analysis could bring us far, but at this stage it is very much wishful thinking and has to be numerically tested [19].

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