Complement G_X

EXERCISES

1. Consider a deuterium atom (composed of a nucleus of spin $I = 1$ and an electron). The electronic angular momentum is $\mathbf{J} = \mathbf{L} + \mathbf{S}$, where $\mathbf{L}$ is the orbital angular momentum of the electron and $\mathbf{S}$ is its spin. The total angular momentum of the atom is $\mathbf{F} = \mathbf{J} + \mathbf{I}$, where $\mathbf{I}$ is the nuclear spin. The eigenvalues of $\mathbf{J}^2$ and $\mathbf{F}^2$ are $J(J+1)\hbar^2$ and $F(F+1)\hbar^2$ respectively.

a. What are the possible values of the quantum numbers J and F for a deuterium atom in the 1s ground state?

b. Same question for deuterium in the 2p excited state.

2. The hydrogen atom nucleus is a proton of spin $I = 1/2$.

a. In the notation of the preceding exercise, what are the possible values of the quantum numbers J and F for a hydrogen atom in the 2p level?

b. Let $\{ | n, l, m \rangle \}$ be the stationary states of the Hamiltonian H_0 of the hydrogen atom studied in §C of chapter VII.

Let $\{ | n, l, s, J, M_J \rangle \}$ be the basis obtained by adding $\mathbf{L}$ and $\mathbf{S}$ to form $\mathbf{J}$ ($M_J\hbar$ is the eigenvalue of J_z); and let $\{ | n, l, s, J, I, F, M_F \rangle \}$ be the basis obtained by adding $\mathbf{J}$ and $\mathbf{I}$ to form $\mathbf{F}$ ($M_F\hbar$ is the eigenvalue of F_z).

The magnetic moment operator of the electron is :

$$\mathbf{M} = \mu_B(\mathbf{L} + 2\mathbf{S})/\hbar$$

In each of the subspaces $\mathscr{E}(n = 2, l = 1, s = 1/2, J, I = 1/2, F)$ arising from the 2p level and subtended by the $2F + 1$ vectors

$$| n = 2, l = 1, s = \frac{1}{2}, J, I = \frac{1}{2}, F, M_F \rangle$$

corresponding to fixed values of J and F, the projection theorem (*cf.* complement D_X, §§ 2-c and 3) enables us to write:

$$\mathbf{M} = g_{JF}\, \mu_B \mathbf{F}/\hbar$$

Calculate the various possible values of the Landé factors g_{JF} corresponding to the 2p level.

3. Consider a system composed of two spin 1/2 particles whose orbital variables are ignored. The Hamiltonian of the system is:

$$H = \omega_1 S_{1z} + \omega_2 S_{2z}$$

where S_{1z} and S_{2z} are the projections of the spins $\mathbf{S}_1$ and $\mathbf{S}_2$ of the two particles onto Oz, and ω_1 and ω_2 are real constants.

a. The initial state of the system, at time $t = 0$, is:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}[|+-\rangle + |-+\rangle]$$

(with the notation of § B of chapter X). At time t, $\mathbf{S}^2 = (\mathbf{S}_1 + \mathbf{S}_2)^2$ is measured. What results can be found, and with what probabilities?

b. If the initial state of the system is arbitrary, what Bohr frequencies can appear in the evolution of $\langle \mathbf{S}^2 \rangle$? Same question for $S_x = S_{1x} + S_{2x}$.

4. Consider a particle (*a*) of spin 3/2 which can disintegrate into two particles, (*b*) of spin 1/2 and (*c*) of spin 0. We place ourselves in the rest frame of (*a*). Total angular momentum is conserved during the disintegration.

a. What values can be taken on by the relative orbital angular momentum of the two final particles? Show that there is only one possible value if the parity of the relative orbital state is fixed. Would this result remain valid if the spin of particle (*a*) were greater than 3/2?

b. Assume that particle (*a*) is initially in the spin state characterized by the eigenvalue $m_a\hbar$ of its spin component along Oz. We know that the final orbital state has a definite parity. Is it possible to determine this parity by measuring the probabilities of finding particle (*b*) either in the state $|+\rangle$ or in the state $|-\rangle$ (you may use the general formulas of complement A_X, §2)?

5. Let $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3$ be the total angular momentum of three spin 1/2 particles (whose orbital variables will be ignored). Let $|\varepsilon_1, \varepsilon_2, \varepsilon_3\rangle$ be the eigenstates common to S_{1z}, S_{2z}, S_{3z}, of respective eigenvalues $\varepsilon_1\hbar/2$, $\varepsilon_2\hbar/2$, $\varepsilon_3\hbar/2$. Give a basis of eigenvectors common to $\mathbf{S}^2$ and S_z, in terms of the kets $|\varepsilon_1, \varepsilon_2, \varepsilon_3\rangle$. Do these two operators form a C.S.C.O.? (Begin by adding two of the spins, then add the partial angular momentum so obtained to the third one.)

6. Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be the intrinsic angular momenta of two spin 1/2 particles, $\mathbf{R}_1$ and $\mathbf{R}_2$, their position observables, and m_1 and m_2, their masses $\left(\text{with } \mu = \frac{m_1 m_2}{m_1 + m_2}, \text{ the reduced mass}\right)$. Assume that the interaction W between the two particles is of the form:

$$W = U(R) + V(R)\frac{\mathbf{S}_1 \cdot \mathbf{S}_2}{\hbar^2}$$

where $U(R)$ and $V(R)$ depend only on the distance $R = |\mathbf{R}_1 - \mathbf{R}_2|$ between the particles.

a. Let $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ be the total spin of the two particles.

α. Show that:

$$P_1 = \frac{3}{4} + \frac{\mathbf{S}_1 \cdot \mathbf{S}_2}{\hbar^2}$$

$$P_0 = \frac{1}{4} - \frac{\mathbf{S}_1 \cdot \mathbf{S}_2}{\hbar^2}$$

are the projectors onto the total spin states $S = 1$ and $S = 0$ respectively.

β. Show from this that $W = W_1(R)P_1 + W_0(R)P_0$, where $W_1(R)$ and $W_0(R)$ are two functions of R, to be expressed in terms of $U(R)$ and $V(R)$.

b. Write the Hamiltonian H of the "relative particle" in the center of mass frame; $\mathbf{P}$ denotes the momentum of this relative particle. Show that H commutes with $\mathbf{S}^2$ and does not depend on S_z. Show from this that it is possible to study separately the eigenstates of H corresponding to $S = 1$ and $S = 0$.

Show that one can find eigenstates of H, with eigenvalue E, of the form:

$$|\psi_E\rangle = \lambda_{00}\,|\varphi_E^0\rangle\,|S = 0, M = 0\rangle + \sum_{M=-1}^{+1} \lambda_{1M}\,|\varphi_E^1\rangle\,|S = 1, M\rangle$$

where λ_{00} and λ_{1M} are constants, and $|\varphi_E^0\rangle$ and $|\varphi_E^1\rangle$ are kets of the state space $\mathscr{E}_r$ of the relative particle ($M\hbar$ is the eigenvalue of S_z). Write the eigenvalue equations satisfied by $|\varphi_E^0\rangle$ and $|\varphi_E^1\rangle$.

c. We want to study collisions between the two particles under consideration. Let $E = \hbar^2k^2/2\mu$ be the energy of the system in the center of mass frame. We assume in all that follows that, before the collision, one of the particles is in the $|+\rangle$ spin state, and the other one, in the $|-\rangle$ spin state. Let $|\psi_k^{\uparrow\downarrow}\rangle$ be the corresponding stationary scattering state (*cf.* chap. VIII, § B). Show that:

$$|\psi_k^{\uparrow\downarrow}\rangle = \frac{1}{\sqrt{2}}|\varphi_k^0\rangle\,|S = 0, M = 0\rangle + \frac{1}{\sqrt{2}}|\varphi_k^1\rangle\,|S = 1, M = 0\rangle$$

where $|\varphi_k^0\rangle$ and $|\varphi_k^1\rangle$ are the stationary scattering states for a spinless particle of mass μ, scattered respectively by the potentials $W_0(R)$ and $W_1(R)$.

d. Let $f_0(\theta)$ and $f_1(\theta)$ be the scattering amplitudes which correspond to $|\varphi_k^0\rangle$ and $|\varphi_k^1\rangle$. Calculate, in terms of $f_0(\theta)$ and $f_1(\theta)$, the scattering cross section $\sigma_b(\theta)$ of the two particles in the θ direction, with simultaneous flip of the two spins (the spin which was in the $|+\rangle$ state goes into the $|-\rangle$ state, and vice versa).

e. Let δ_l^0 and δ_l^1 be the phase shifts of the l partial waves associated respectively with $W_0(R)$ and $W_1(R)$ (*cf.* chap. VIII, § C-3). Show that the total scattering cross section σ_b, with simultaneous flip of the two spins, is equal to:

$$\sigma_b = \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l + 1)\sin^2(\delta_l^1 - \delta_l^0)$$

7. We define the standard components of a vector operator **V** as the three operators:

$$\begin{cases} V_1^{(1)} = -\dfrac{1}{\sqrt{2}}(V_x + iV_y) \\ V_0^{(1)} = V_z \\ V_{-1}^{(1)} = \dfrac{1}{\sqrt{2}}(V_x - iV_y) \end{cases}$$

Using the standard components $V_p^{(1)}$ and $W_q^{(1)}$ of the two vector operators **V** and **W**, we construct the operators:

$$[V^{(1)} \otimes W^{(1)}]_M^{(K)} = \sum_p \sum_q \langle 1, 1 ; p, q \mid K, M \rangle \, V_p^{(1)} W_q^{(1)}$$

where the $\langle 1, 1 ; p, q \mid K, M \rangle$ are the Clebsch-Gordan coefficients entering into the addition of two angular momenta 1 (these coefficients can be obtained from the results of § 1 of complement A_X).

a. Show that $[V^{(1)} \otimes W^{(1)}]_0^{(0)}$ is proportional to the scalar product $\mathbf{V} \cdot \mathbf{W}$ of the two vector operators.

b. Show that the three operators $[V^{(1)} \otimes W^{(1)}]_M^{(1)}$ are proportional to the three standard components of the vector operator $\mathbf{V} \times \mathbf{W}$.

c. Express the five components $[V^{(1)} \otimes W^{(1)}]_M^{(2)}$ in terms of V_z, $V_\pm = V_x \pm iV_y$, W_z, $W_\pm = W_x \pm iW_y$.

d. We choose $\mathbf{V} = \mathbf{W} = \mathbf{R}$, where **R** is the position observable of a particle. Show that the five operators $[R^{(1)} \otimes R^{(1)}]_M^{(2)}$ are proportional to the five components Q_2^M of the electric quadrupole moment operator of this particle [*cf.* formula (29) of complement E_X].

e. We choose $\mathbf{V} = \mathbf{W} = \mathbf{L}$, where **L** is the orbital angular momentum of the particle. Express the five operators $[L^{(1)} \otimes L^{(1)}]_M^{(2)}$ in terms of L_z, L_+, L_-. What are the selection rules satisfied by these five operators in a standard basis $\{ \mid k, l, m \rangle \}$ of eigenstates common to $\mathbf{L}^2$ and L_z (in other words, on what conditions is the matrix element

$$\langle k, l, m \mid [L^{(1)} \otimes L^{(1)}]_M^{(2)} \mid k', l', m' \rangle$$

non-zero)?

8. Irreducible tensor operators ; Wigner-Eckart theorem

The $2K + 1$ operators $T_Q^{(K)}$, with K an integer $\geqslant 0$ and

$Q = -K, -K + 1, \ldots, +K,$

are, by definition, the $2K + 1$ components of an irreducible tensor operator of rank K if they satisfy the following commutation relations with the total angular momentum **J** of the physical system:

$$[J_z, T_Q^{(K)}] = \hbar Q T_Q^{(K)} \tag{1}$$

$$[J_+, T_Q^{(K)}] = \hbar\sqrt{K(K+1) - Q(Q+1)}\, T_{Q+1}^{(K)} \tag{2}$$

$$[J_-, T_Q^{(K)}] = \hbar\sqrt{K(K+1) - Q(Q-1)}\, T_{Q-1}^{(K)} \tag{3}$$

a. Show that a scalar operator is an irreducible tensor operator of rank $K = 0$, and that the three standard components of a vector operator (*cf.* exercise 7) are the components of an irreducible tensor operator of rank $K = 1$.

b. Let $\{ | k, J, M \rangle \}$ be a standard basis of common eigenstates of $\mathbf{J}^2$ and J_z. By taking both sides of (1) to have the same matrix elements between $| k, J, M \rangle$ and $| k', J', M' \rangle$, show that $\langle k, J, M | T_Q^{(K)} | k', J', M' \rangle$ is zero if M is not equal to $Q + M'$.

c. Proceeding in the same way with relations (2) and (3), show that the $(2J+1)(2K+1)(2J'+1)$ matrix elements $\langle k, J, M | T_Q^{(K)} | k', J', M' \rangle$ corresponding to fixed values of k, J, K, k', J' satisfy recurrence relations identical to those satisfied by the $(2J+1)(2K+1)(2J'+1)$ Clebsch-Gordan coefficients $\langle J', K; M', Q | J, M \rangle$ (*cf.* complement B_X, §§ 1-c and 2) corresponding to fixed values of J, K, J'.

d. Show that:

$$\langle k, J, M | T_Q^{(K)} | k', J', M' \rangle = \alpha \langle J', K; M', Q | J, M \rangle \tag{4}$$

where α is a constant depending only on k, J, K, k', J', which is usually written in the form:

$$\alpha = \frac{1}{\sqrt{2J+1}} \langle k, J \| T^{(K)} \| k', J' \rangle$$

e. Show that, conversely, if $(2K+1)$ operators $T_Q^{(K)}$ satisfy relation (4) for all $| k, J, M \rangle$ and $| k', J', M' \rangle$, they satisfy relations (1), (2) and (3), that is, they constitute the $(2K+1)$ components of an irreducible tensor operator of rank K.

f. Show that, for a spinless particle, the electric multipole moment operators Q_l^m introduced in complement E_X are irreducible tensor operators of rank l in the state space $\mathscr{E}_\mathbf{r}$ of this particle. Show that, in addition, when the spin degrees of freedom are taken into account, the operators Q_l^m remain irreducible tensor operators in the state space $\mathscr{E}_\mathbf{r} \otimes \mathscr{E}_s$ (where $\mathscr{E}_s$ is the spin state space).

g. Derive the selection rules satisfied by the Q_l^m in a standard basis $\{ | k, l, J, M_J \rangle \}$ obtained by adding the orbital angular momentum $\mathbf{L}$ and the spin $\mathbf{S}$ of the particle to form the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ [$l(l+1)\hbar^2$, $J(J+1)\hbar^2$, $M_J\hbar$ are the eigenvalues of $\mathbf{L}^2$, $\mathbf{J}^2$, J_z respectively].

9. Let $A_{Q_1}^{(K_1)}$ be an irreducible tensor operator (exercise 8) of rank K_1 acting in a state space $\mathscr{E}_1$, and $B_{Q_2}^{(K_2)}$, an irreducible tensor operator of rank K_2 acting in a state space $\mathscr{E}_2$. With $A_{Q_1}^{(K_1)}$ and $B_{Q_2}^{(K_2)}$, we construct the operator :

$$C_Q^{(K)} = [A^{(K_1)} \otimes B^{(K_2)}]_Q^{(K)} = \sum_{Q_1 Q_2} \langle K_1, K_2 ; Q_1, Q_2 | K, Q \rangle A_{Q_1}^{(K_1)} B_{Q_2}^{(K_2)}$$

a. Using the recurrence relations for Clebsch-Gordan coefficients (*cf.* complement B_X), show that the $C_Q^{(K)}$ satisfy commutation relations (1), (2) and (3) of exercise 8 with the total angular momentum $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ of the system. Show that the $C_Q^{(K)}$ are the components of an irreducible tensor operator of rank K.

b. Show that the operator $\sum_Q (-1)^Q A_Q^{(K)} B_{-Q}^{(K)}$ is a scalar operator (you may use the results of §3-d of complement B_X).

10. Addition of three angular momenta

Let $\mathscr{E}(1)$, $\mathscr{E}(2)$, $\mathscr{E}(3)$ be the state spaces of three systems, (1), (2) and (3), of angular momenta $\mathbf{J}_1$, $\mathbf{J}_2$, $\mathbf{J}_3$. We shall write $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3$ for the total angular momentum. Let $\{ | k_a, j_a, m_a \rangle \}$, $\{ | k_b, j_b, m_b \rangle \}$, $\{ | k_c, j_c, m_c \rangle \}$ be the standard bases of $\mathscr{E}(1)$, $\mathscr{E}(2)$, $\mathscr{E}(3)$, respectively. To simplify the notation, we shall omit the indices k_a, k_b, k_c, as we did in chapter X.

We are interested in the eigenstates and eigenvalues of the total angular momentum in the subspace $\mathscr{E}(j_a, j_b, j_c)$ subtended by the kets:

$$\{ | j_a m_a \rangle | j_b m_b \rangle | j_c m_c \rangle \}$$
$$- j_a \leqslant m_a \leqslant j_a, \; - j_b \leqslant m_b \leqslant j_b, \; - j_c \leqslant m_c \leqslant j_c \tag{1}$$

We want to add j_a, j_b, j_c to form an eigenstate of $\mathbf{J}^2$ and J_z characterized by the quantum numbers j_f and m_f. We shall denote by:

$$| j_a, (j_b j_c) j_e ; j_f m_f \rangle \tag{2}$$

such a normalized eigenstate obtained by first adding j_b to j_c to form an angular momentum j_e, then adding j_a to j_e to form the state $| j_f m_f \rangle$. One could also add j_a and j_b to form j_g and then add j_c to j_g to form the normalized state $| j_f m_f \rangle$, written:

$$| (j_a j_b) j_g, j_c ; j_f m_f \rangle \tag{3}$$

a. Show that the system of kets (2), corresponding to the various possible values of j_e, j_f, m_f, forms an orthonormal basis in $\mathscr{E}(j_a, j_b, j_c)$. Same question for the system of kets (3), corresponding to the various values of j_g, j_f, m_f.

b. Show, by using the operators $J_\pm$, that the scalar product $\langle (j_a j_b) j_g, j_c ; j_f m_f | j_a, (j_b j_c) j_e ; j_f m_f \rangle$ does not depend on m_f, denoting such a scalar product by $\langle (j_a j_b) j_g, j_c ; j_f | j_a, (j_b j_c) j_e ; j_f \rangle$.

c. Show that:

$$| j_a, (j_b j_c) j_e ; j_f m_f \rangle = \sum_{j_g} \langle (j_a j_b) j_g, j_c ; j_f | j_a, (j_b j_c) j_e ; j_f \rangle \, | (j_a j_b) j_g, j_c ; j_f m_f \rangle \tag{4}$$

d. Using the Clebsch-Gordan coefficients, write the expansions for vectors (2) and (3) in the basis (1). Show that:

$$\sum_{m_e} \langle j_b, j_c ; m_b, m_c \mid j_e, m_e \rangle \langle j_a, j_e ; m_a, m_e \mid j_f, m_f \rangle =$$

$$\sum_{j_g m_g} \langle j_a, j_b ; m_a, m_b \mid j_g, m_g \rangle \langle j_g, j_c ; m_g, m_c \mid j_f, m_f \rangle$$

$$\times \langle (j_a j_b) j_g, j_c ; j_f \mid j_a, (j_b j_c) j_e ; j_f \rangle \quad (5)$$

e. Starting with relation (5), prove, using the Clebsch-Gordan coefficient orthogonality relations, the following relations:

$$\sum_{m_a m_b m_e} \langle j_b, j_c ; m_b, m_c \mid j_e, m_e \rangle \langle j_a, j_e ; m_a, m_e \mid j_f, m_f \rangle \langle j_d, m_d \mid j_a, j_b ; m_a, m_b \rangle$$

$$= \langle j_d, j_c ; m_d, m_c \mid j_f, m_f \rangle \langle (j_a j_b) j_d, j_c ; j_f \mid j_a, (j_b j_c) j_e ; j_f \rangle \quad (6)$$

as well as:

$$\langle (j_a j_b) j_d, j_c ; j_f \mid j_a, (j_b j_c) j_e ; j_f \rangle = \frac{1}{2j_f + 1} \sum_{m_a m_b m_c m_d m_e m_f} \langle j_b, j_c ; m_b, m_c \mid j_e, m_e \rangle$$

$$\times \langle j_a, j_e ; m_a, m_e \mid j_f, m_f \rangle \langle j_d, m_d \mid j_a, j_b ; m_a, m_b \rangle \langle j_f, m_f \mid j_d, j_c ; m_d, m_c \rangle \quad (7)$$

Exercises 8 and 9:

References : see references of complement D_X

Exercise 10:

References: Edmonds (2.21), chap. 6; Messiah (1.17), §XIII-29 and App. C; Rose (2.19), App. I.