Complement H_{II}

EXERCISES

Dirac notation. Commutators. Eigenvectors and eigenvalues

1. $| \varphi_n \rangle$ are the eigenstates of a Hermitian operator H (H is, for example, the Hamiltonian of an arbitrary physical system). Assume that the states $| \varphi_n \rangle$ form a discrete orthonormal basis. The operator $U(m, n)$ is defined by:

$$U(m, n) = | \varphi_m \rangle \langle \varphi_n |$$

a. Calculate the adjoint $U^\dagger(m, n)$ of $U(m, n)$.

b. Calculate the commutator $[H, U(m, n)]$.

c. Prove the relation:

$$U(m, n)U^\dagger(p, q) = \delta_{nq} U(m, p)$$

d. Calculate Tr $\{ U(m, n) \}$, the trace of the operator $U(m, n)$.

e. Let A be an operator, with matrix elements $A_{mn} = \langle \varphi_m | A | \varphi_n \rangle$. Prove the relation:

$$A = \sum_{m,n} A_{mn} \, U(m, n)$$

f. Show that $A_{pq} = \text{Tr} \{ A U^\dagger(p, q) \}$.

2. In a two-dimensional vector space, consider the operator whose matrix, in an orthonormal basis $\{ | 1 \rangle, | 2 \rangle \}$, is written:

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

a. Is σ_y Hermitian? Calculate its eigenvalues and eigenvectors (giving their normalized expansion in terms of the $\{ | 1 \rangle, | 2 \rangle \}$ basis).

b. Calculate the matrices which represent the projectors onto these eigenvectors. Then verify that they satisfy the orthogonality and closure relations.

c. Same questions for the matrices:

$$M = \begin{pmatrix} 2 & i\sqrt{2} \\ -i\sqrt{2} & 3 \end{pmatrix}$$

and, in a three-dimensional space

$$L_y = \frac{\hbar}{2i} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}$$

3. The state space of a certain physical system is three-dimensional. Let $\{ | u_1 \rangle, | u_2 \rangle, | u_3 \rangle \}$ be an orthonormal basis of this space. The kets $| \psi_0 \rangle$ and $| \psi_1 \rangle$ are defined by:

$$| \psi_0 \rangle = \frac{1}{\sqrt{2}} | u_1 \rangle + \frac{i}{2} | u_2 \rangle + \frac{1}{2} | u_3 \rangle$$

$$| \psi_1 \rangle = \frac{1}{\sqrt{3}} | u_1 \rangle + \frac{i}{\sqrt{3}} | u_3 \rangle$$

a. Are these kets normalized?

b. Calculate the matrices ρ_0 and ρ_1 representing, in the $\{ | u_1 \rangle, | u_2 \rangle, | u_3 \rangle \}$ basis, the projection operators onto the state $| \psi_0 \rangle$ and onto the state $| \psi_1 \rangle$. Verify that these matrices are Hermitian.

4. Let K be the operator defined by $K = | \varphi \rangle \langle \psi |$, where $| \varphi \rangle$ and $| \psi \rangle$ are two vectors of the state space.

a. Under what condition is K Hermitian?

b. Calculate K^2. Under what condition is K a projector?

c. Show that K can always be written in the form $K = \lambda P_1 P_2$ where λ is a constant to be calculated and P_1 and P_2 are projectors.

5. Let P_1 be the orthogonal projector onto the subspace $\mathscr{E}_1$, P_2 the orthogonal projector onto the subspace $\mathscr{E}_2$. Show that, for the product $P_1 P_2$ to be an orthogonal projector as well, it is necessary and sufficient that P_1 and P_2 commute. In this case, what is the subspace onto which $P_1 P_2$ projects?

6. The σ_x matrix is defined by:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Prove the relation:

$$e^{i\alpha\sigma_x} = I \cos \alpha + i\sigma_x \sin \alpha$$

where I is the 2×2 unit matrix.

7. Establish, for the σ_y matrix given in exercise 2, a relation analogous to the one proved for σ_x in the preceding exercise. Generalize for all matrices of the form:

$$\sigma_u = \lambda \sigma_x + \mu \sigma_y$$

with:

$$\lambda^2 + \mu^2 = 1$$

Calculate the matrices representing $e^{2i\sigma_x}$, $(e^{i\sigma_x})^2$ and $e^{i(\sigma_x+\sigma_y)}$. Is $e^{2i\sigma_x}$ equal to $(e^{i\sigma_x})^2$? $e^{i(\sigma_x+\sigma_y)}$ to $e^{i\sigma_x}\, e^{i\sigma_y}$?

8. Consider the Hamiltonian H of a particle in a one-dimensional problem defined by:

$$H = \frac{1}{2m} P^2 + V(X)$$

where X and P are the operators defined in § E of chapter II and which satisfy the relation: $[X, P] = i\hbar$. The eigenvectors of H are denoted by $|\varphi_n\rangle$: $H|\varphi_n\rangle = E_n|\varphi_n\rangle$, where n is a discrete index.

a. Show that:

$$\langle \varphi_n | P | \varphi_{n'} \rangle = \alpha \langle \varphi_n | X | \varphi_{n'} \rangle$$

where α is a coefficient which depends on the difference between E_n and $E_{n'}$. Calculate α (hint: consider the commutator $[X, H]$).

b. From this, deduce, using the closure relation, the equation:

$$\sum_{n'} (E_n - E_{n'})^2 |\langle \varphi_n | X | \varphi_{n'} \rangle|^2 = \frac{\hbar^2}{m^2} \langle \varphi_n | P^2 | \varphi_n \rangle$$

9. Let H be the Hamiltonian operator of a physical system. Denote by $|\varphi_n\rangle$ the eigenvectors of H, with eigenvalues E_n:

$$H|\varphi_n\rangle = E_n|\varphi_n\rangle$$

a. For an arbitrary operator A, prove the relation:

$$\langle \varphi_n | [A, H] | \varphi_n \rangle = 0.$$

b. Consider a one-dimensional problem, where the physical system is a particle of mass m and of potential energy $V(X)$. In this case, H is written:

$$H = \frac{1}{2m} P^2 + V(X)$$

α. In terms of P, X and $V(X)$, find the commutators: $[H, P]$, $[H, X]$ and $[H, XP]$.

β. Show that the matrix element $\langle \varphi_n | P | \varphi_n \rangle$ (which we shall interpret in chapter III as the mean value of the momentum in the state $|\varphi_n\rangle$) is zero.

γ. Establish a relation between $E_k = \langle \varphi_n | \frac{P^2}{2m} | \varphi_n \rangle$ (the mean value of the kinetic energy in the state $|\varphi_n\rangle$) and $\langle \varphi_n | X \frac{dV}{dX} | \varphi_n \rangle$. Since the mean value of the potential energy in the state $|\varphi_n\rangle$ is $\langle \varphi_n | V(x) | \varphi_n \rangle$, how is it related to the mean value of the kinetic energy when:

$$V(X) = V_0 X^\lambda$$

$(\lambda = 2, 4, 6 ...; V_0 > 0)$?

10 Using the relation $\langle x \mid p \rangle = (2\pi\hbar)^{-1/2} e^{ipx/\hbar}$, find the expressions $\langle x \mid XP \mid \psi \rangle$ and $\langle x \mid PX \mid \psi \rangle$ in terms of $\psi(x)$. Can these results be found directly by using the fact that in the $\{ \mid x \rangle \}$ representation, P acts like $\frac{\hbar}{i}\frac{d}{dx}$?

Sets of commuting observables and C.S.C.O.'S

11. Consider a physical system whose three-dimensional state space is spanned by the orthonormal basis formed by the three kets $| u_1 \rangle$, $| u_2 \rangle$, $| u_3 \rangle$. In the basis of these three vectors, taken in this order, the two operators H and B are defined by:

$$H = \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad B = b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where ω_0 and b are real constants.

a. Are H and B Hermitian ?

b. Show that H and B commute. Give a basis of eigenvectors common to H and B.

c. Of the sets of operators: $\{ H \}$, $\{ B \}$, $\{ H, B \}$, $\{ H^2, B \}$, which form a C.S.C.O. ?

12. In the same state space as that of the preceding exercise, consider two operators L_z and S defined by:

$$L_z | u_1 \rangle = | u_1 \rangle \qquad L_z | u_2 \rangle = 0 \qquad L_z | u_3 \rangle = - | u_3 \rangle$$
$$S | u_1 \rangle = | u_3 \rangle \qquad S | u_2 \rangle = | u_2 \rangle \qquad S | u_3 \rangle = | u_1 \rangle$$

a. Write the matrices which represent, in the $\{ | u_1 \rangle, | u_2 \rangle, | u_3 \rangle \}$ basis, the operators L_z, L_z^2, S, S^2. Are these operators observables ?

b. Give the form of the most general matrix which represents an operator which commutes with L_z. Same question for L_z^2, then for S^2.

c. Do L_z^2 and S form a C.S.C.O. ? Give a basis of common eigenvectors.