Complement M_V

EXERCISES

1. Consider a harmonic oscillator of mass m and angular frequency ω. At time $t = 0$, the state of this oscillator is given by:

$$| \psi(0) \rangle = \sum_n c_n | \varphi_n \rangle$$

where the states $| \varphi_n \rangle$ are stationary states with energies $(n + 1/2)\hbar\omega$.

a. What is the probability $\mathscr{P}$ that a measurement of the oscillator's energy performed at an arbitrary time $t > 0$, will yield a result greater than $2\hbar\omega$? When $\mathscr{P} = 0$, what are the non-zero coefficients c_n?

b. From now on, assume that only c_0 and c_1 are different from zero. Write the normalization condition for $|\psi(0)\rangle$ and the mean value $\langle H \rangle$ of the energy in terms of c_0 and c_1. With the additional requirement $\langle H \rangle = \hbar\omega$, calculate $|c_0|^2$ and $|c_1|^2$.

c. As the normalized state vector $|\psi(0)\rangle$ is defined only to within a global phase factor, we fix this factor by choosing c_0 real and positive. We set: $c_1 = |c_1| e^{i\theta_1}$. We assume that $\langle H \rangle = \hbar\omega$ and that:

$$\langle X \rangle = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}$$

Calculate θ_1.

d. With $|\psi(0)\rangle$ so determined, write $|\psi(t)\rangle$ for $t > 0$ and calculate the value of θ_1 at t. Deduce the mean value $\langle X \rangle(t)$ of the position at t.

2. Anisotropic three-dimensional harmonic oscillator

In a three-dimensional problem, consider a particle of mass m and of potential energy:

$$V(X, Y, Z) = \frac{m\omega^2}{2}\left[\left(1 + \frac{2\lambda}{3}\right)(X^2 + Y^2) + \left(1 - \frac{4\lambda}{3}\right)Z^2\right]$$

where ω and λ are constants which satisfy:

$$\omega \geqslant 0 \quad , \quad 0 \leqslant \lambda < \frac{3}{4}$$

a. What are the eigenstates of the Hamiltonian and the corresponding energies?

b. Calculate and discuss, as functions of λ, the variation of the energy, the parity and the degree of degeneracy of the ground state and the first two excited states.

3. Harmonic oscillator: two particles

Two particles of the same mass m, with positions X_1 and X_2 and momenta P_1 and P_2, are subject to the same potential:

$$V(X) = \frac{1}{2} m\omega^2 X^2$$

The two particles do not interact.

a. Write the operator H, the Hamiltonian of the two-particle system. Show that H can be written:

$$H = H_1 + H_2$$

where H_1 and H_2 act respectively only in the state space of particle (1) and in that of particle (2). Calculate the energies of the two-particle system, their degrees of degeneracy, and the corresponding wave functions.

b. Does H form a C.S.C.O.? Same question for the set $\{ H_1, H_2 \}$. We denote by $| \Phi_{n_1,n_2} \rangle$ the eigenvectors common to H_1 and H_2. Write the orthonormalization and closure relations for the states $| \Phi_{n_1,n_2} \rangle$.

c. Consider a system which, at $t = 0$, is in the state :

$$| \psi(0) \rangle = \frac{1}{2} (| \Phi_{0,0} \rangle + | \Phi_{1,0} \rangle + | \Phi_{0,1} \rangle + | \Phi_{1,1} \rangle)$$

What results can be found, and with what probabilities, if at this time one measures:

– the total energy of the system?
– the energy of particle (1)?
– the position or velocity of this particle?

4. (This exercise is a continuation of the preceding one and uses the same notation.)

The two-particle system, at $t = 0$, is in the state $| \psi(0) \rangle$ given in exercise 3.

a. At $t = 0$, one measures the total energy H and one finds the result $2\hbar\omega$.

α. Calculate the mean values of the position, the momentum, and the energy of particle (1) at an arbitrary positive t. Same question for particle (2).

β. At $t > 0$, one measures the energy of particle (1). What results can be found, and with what probabilities? Same question for a measurement of the position of particle (1); trace the curve for the corresponding probability density.

b. Instead of measuring the total energy H, at $t = 0$, one measures the energy H_2 of particle (2); the result obtained is $\hbar\omega/2$. What happens to the answers to questions α and β of *a*?

5. (This exercise is a continuation of exercise 3 and uses the same notation.) We denote by $|\Phi_{n_1,n_2}\rangle$ the eigenstates common to H_1 and H_2, of eigenvalues $(n_1 + 1/2)\hbar\omega$ and $(n_2 + 1/2)\hbar\omega$. The "two particle exchange" operator P_e is defined by:

$$P_e \,|\Phi_{n_1,n_2}\rangle = |\Phi_{n_2,n_1}\rangle$$

a. Prove that $P_e^{-1} = P_e$ and that P_e is unitary. What are the eigenvalues of P_e ? Let $B' = P_e B P_e^\dagger$ be the observable resulting from the transformation by P_e of an arbitrary observable B. Show that the condition $B' = B$ (B invariant under exchange of the two particles) is equivalent to $[B, P_e] = 0$.

b. Show that:

$$P_e H_1 P_e^\dagger = H_2$$
$$P_e H_2 P_e^\dagger = H_1$$

Does H commute with P_e ? Calculate the action of P_e on the observables X_1, P_1, X_2, P_2.

c. Construct a basis of eigenvectors common to H and P_e. Do these two operators form a C.S.C.O.? What happens to the spectrum of H and the degeneracy of its eigenvalues if one retains only the eigenvectors $|\Phi\rangle$ of H for which $P_e|\Phi\rangle = -|\Phi\rangle$?

6. Charged harmonic oscillator in a variable electric field

A one-dimensional harmonic oscillator is composed of a particle of mass m, charge q and potential energy $V(X) = \frac{1}{2} m\omega^2 X^2$. We assume in this exercise that the particle is placed in an electric field $\mathscr{E}(t)$ parallel to Ox and time-dependent, so that to $V(x)$ must be added the potential energy:

$$W(t) = -\,q\mathscr{E}(t)X$$

a. Write the Hamiltonian $H(t)$ of the particle in terms of the operators a and $a^\dagger$. Calculate the commutators of a and $a^\dagger$ with $H(t)$.

b. Let $\alpha(t)$ be the number defined by:

$$\alpha(t) = \langle \psi(t) | a | \psi(t) \rangle$$

where $|\psi(t)\rangle$ is the normalized state vector of the particle under study. Show from the results of the preceding question that $\alpha(t)$ satisfies the differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t) = -\,i\omega\,\alpha(t) + i\lambda(t)$$

where $\lambda(t)$ is defined by:

$$\lambda(t) = \frac{q}{\sqrt{2m\hbar\omega}}\,\mathscr{E}(t)$$

Integrate this differential equation. At time t, what are the mean values of the position and momentum of the particle?

c. The ket $|\,\varphi(t)\rangle$ is defined by:

$$|\,\varphi(t)\rangle = [a - \alpha(t)]\,|\,\psi(t)\rangle$$

where $\alpha(t)$ has the value calculated in *b*. Using the results of questions *a* and *b*, show that the evolution of $|\,\varphi(t)\rangle$ is given by:

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\,\varphi(t)\rangle = [H(t) + \hbar\omega]\,|\,\varphi(t)\rangle$$

How does the norm of $|\,\varphi(t)\rangle$ vary with time?

d. Assuming that $|\,\psi(0)\rangle$ is an eigenvector of a with the eigenvalue $\alpha(0)$, show that $|\,\psi(t)\rangle$ is also an eigenvector of a, and calculate its eigenvalue.

Find at time t the mean value of the unperturbed Hamiltonian

$$H_0 = H(t) - W(t)$$

as a function of $\alpha(0)$. Give the root-mean-square deviations ΔX, ΔP and ΔH_0; how do they vary with time?

e. Assume that at $t = 0$, the oscillator is in the ground state $|\,\varphi_0\rangle$. The electric field acts between times 0 and T and then falls to zero. When $t > T$, what is the evolution of the mean values $\langle X\rangle(t)$ and $\langle P\rangle(t)$? Application : assume that between 0 and T, the field $\mathscr{E}(t)$ is given by $\mathscr{E}(t) = \mathscr{E}_0 \cos(\omega' t)$; discuss the phenomena observed (resonance) in terms of $\Delta\omega = \omega' - \omega$. If, at $t > T$, the energy is measured, what results can be found, and with what probabilities?

7. Consider a one-dimensional harmonic oscillator of Hamiltonian H and stationary states $|\,\varphi_n\rangle$:

$$H\,|\,\varphi_n\rangle = (n + 1/2)\hbar\omega\,|\,\varphi_n\rangle$$

The operator $U(k)$ is defined by:

$$U(k) = \mathrm{e}^{ikX}$$

where k is real.

a. Is $U(k)$ unitary? Show that, for all n, its matrix elements satisfy the relation:

$$\sum_{n'} |\langle\,\varphi_n\,|\,U(k)\,|\,\varphi_{n'}\rangle|^2 = 1$$

b. Express $U(k)$ in terms of the operators a and $a^\dagger$. Use Glauber's formula [formula (63) of complement $\mathrm{B_{II}}$] to put $U(k)$ in the form of a product of exponential operators.

c. Establish the relations:

$$e^{\lambda a} \mid \varphi_0 \rangle = \mid \varphi_0 \rangle$$

$$\langle \varphi_n \mid e^{\lambda a^\dagger} \mid \varphi_0 \rangle = \frac{\lambda^n}{\sqrt{n!}}$$

where λ is an arbitrary complex parameter.

d. Find the expression, in terms of $E_k = \hbar^2 k^2/2m$ and $E_\omega = \hbar\omega$, for the matrix element:

$$\langle \varphi_0 \mid U(k) \mid \varphi_n \rangle$$

What happens when k approaches zero? Could this result have been predicted directly?

8. The evolution operator $U(t, 0)$ of a one-dimensional harmonic oscillator is written:

$$U(t, 0) = e^{-iHt/\hbar}$$

with:

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

a. Consider the operators:

$$\tilde{a}(t) = U^\dagger(t, 0)\, a\, U(t, 0)$$

$$\tilde{a}^\dagger(t) = U^\dagger(t, 0)\, a^\dagger\, U(t, 0)$$

By calculating their action on the eigenkets $\mid \varphi_n \rangle$ of H, find the expression for $\tilde{a}(t)$ and $\tilde{a}^\dagger(t)$ in terms of a and $a^\dagger$.

b. Calculate the operators $\tilde{X}(t)$ and $\tilde{P}(t)$ obtained from X and P by the unitary transformation:

$$\tilde{X}(t) = U^\dagger(t, 0)\, X\, U(t, 0)$$

$$\tilde{P}(t) = U^\dagger(t, 0)\, P\, U(t, 0)$$

How can the relations so obtained be interpreted?

c. Show that $U^\dagger\left(\frac{\pi}{2\omega}, 0\right) \mid x \rangle$ is an eigenvector of P and specify its eigenvalue.

Similarly, establish that $U^\dagger\left(\frac{\pi}{2\omega}, 0\right) \mid p \rangle$ is an eigenvector of X.

d. At $t = 0$, the wave function of the oscillator is $\psi(x, 0)$. How can one obtain from $\psi(x, 0)$ the wave function of the oscillator at all subsequent times $t_q = q\pi/2\omega$ (where q is a positive integer)?

e. Choose for $\psi(x, 0)$ the wave function $\varphi_n(x)$ associated with a stationary state. From the preceding question derive the relation which must exist between $\varphi_n(x)$ and its Fourier transform $\overline{\varphi}_n(p)$.

f. Describe qualitatively the evolution of the wave function in the following cases:

(*i*) $\psi(x, 0) = e^{ikx}$ where k, real, is given.

(*ii*) $\psi(x, 0) = e^{-\rho x}$ where ρ is real and positive.

(*iii*)

$$\psi(x, 0) \begin{cases} = \dfrac{1}{\sqrt{a}} & \text{if } -\dfrac{a}{2} \leqslant x \leqslant \dfrac{a}{2} \\ = 0 & \text{everywhere else} \end{cases}$$

(*iv*) $\psi(x, 0) = e^{-\rho^2 x^2}$ where ρ is real.