3. Exercises

a. SCATTERING OF THE *p* WAVE BY A HARD SPHERE

We wish to study the phase shift $\delta_1(k)$ produced by a hard sphere on the p wave ($l = 1$). In particular, we want to verify that it becomes negligible compared to $\delta_0(k)$ at low energy.

α. Write the radial equation for the function $u_{k,1}(r)$ for $r > r_0$. Show that its general solution is of the form:

$$u_{k,1}(r) = C\left[\frac{\sin kr}{kr} - \cos kr + a\left(\frac{\cos kr}{kr} + \sin kr\right)\right]$$

where C and a are constants.

β. Show that the definition of $\delta_1(k)$ implies that:

$a = \tan \delta_1(k)$

γ. Determine the constant a from the condition imposed on $u_{k,1}(r)$ at $r = r_0$.

δ. Show that, as k approaches zero, $\delta_1(k)$ behaves like* $(kr_0)^3$, which makes it negligible compared to $\delta_0(k)$.

b. "SQUARE SPHERICAL WELL" : BOUND STATES AND SCATTERING RESONANCES

Consider a central potential $V(r)$ such that :

$$\begin{aligned} V(r) &= -V_0 \quad &\text{for} \quad r < r_0 \\ &= 0 \quad &\text{for} \quad r > r_0 \end{aligned}$$

where V_0 is a positive constant. Set:

$$k_0 = \sqrt{\frac{2\mu V_0}{\hbar^2}}$$

We shall confine ourselves to the study of the s wave ($l = 0$).

α. *Bound states* ($E < 0$)

(*i*) Write the radial equation in the two regions $r > r_0$ and $r < r_0$, as well as the condition at the origin. Show that, if one sets:

$$\rho = \sqrt{\frac{-2\mu E}{\hbar^2}}$$

$$K = \sqrt{k_0^2 - \rho^2}$$

* This result is true in general : for any potential of finite range r_0, the phase shift $\delta_l(k)$ behaves like $(kr_0)^{2l+1}$ at low energies.

the function $u_0(r)$ is necessarily of the form:

$$\begin{aligned} u_0(r) &= A\,e^{-\rho r} \quad &\text{for} \quad & r > r_0 \\ &= B \sin Kr \quad &\text{for} \quad & r < r_0 \end{aligned}$$

(*ii*) Write the matching conditions at $r = r_0$. Deduce from them that the only possible values for ρ are those which satisfy the equation:

$$\tan Kr_0 = -\frac{K}{\rho}$$

(*iii*) Discuss this equation: indicate the number of s bound states as a function of the depth of the well (for fixed r_0) and show, in particular, that there are no bound states if this depth is too small.

β. *Scattering resonances* ($E > 0$)

(*i*) Again write the radial equation, this time setting:

$$k = \sqrt{\frac{2\mu E}{\hbar^2}}$$

$$K' = \sqrt{k_0^2 + k^2}$$

Show that $u_{k,0}(r)$ is of the form:

$$\begin{aligned} u_{k,0}(r) &= A \sin(kr + \delta_0) \quad &\text{for} \quad & r > r_0 \\ &= B \sin K'r \quad &\text{for} \quad & r < r_0 \end{aligned}$$

(*ii*) Choose $A = 1$. Show, using the continuity conditions at $r = r_0$, that the constant B and the phase shift δ_0 are given by:

$$B^2 = \frac{k^2}{k^2 + k_0^2 \cos^2 K'r_0}$$

$$\delta_0 = -kr_0 + \alpha(k)$$

with:

$$\tan \alpha(k) = \frac{k}{K'} \tan K'r_0$$

(*iii*) Trace the curve representing B^2 as a function of k. This curve clearly shows resonances, for which B^2 is maximum. What are the values of k associated with these resonances? What is then the value of $\alpha(k)$? Show that, if there exists such a resonance for a small energy ($kr_0 \ll 1$), the corresponding contribution of the s wave to the total cross section is practically maximal.

γ. *Relation between bound states and scattering resonances*

Assume that $k_0 r_0$ is very close to $(2n + 1)\frac{\pi}{2}$, where n is an integer, and set:

$$k_0 r_0 = (2n + 1)\frac{\pi}{2} + \varepsilon \qquad \text{with} \qquad |\varepsilon| \ll 1$$

(*i*) Show that, if ε is positive, there exists a bound state whose binding energy $E = -\hbar^2\rho^2/2\mu$ is given by:

$$\rho \simeq \varepsilon k_0$$

(*ii*) Show that if, on the other hand, ε is negative, there exists a scattering resonance at energy $E = \hbar^2 k^2/2\mu$ such that:

$$k^2 \simeq -\frac{2k_0\varepsilon}{r_0}$$

(*iii*) Deduce from this that if the depth of the well is gradually decreased (for fixed r_0), the bound state which disappears when $k_0 r_0$ passes through an odd multiple of $\pi/2$ gives rise to a low energy scattering resonance.

References and suggestions for further reading :

Messiah (1.17), chap. IX, § 10 and chap. X, §§ III and IV; Valentin (16.1), Annexe II.