Path Integrals and Coherent States

Marcus A.M. de Aguiar

Dresden, August 2008

Outline

1. Basic formulations

2. Semiclassical limit

3. Applications

Introduction and Motivations

- Why Path Integrals?
- Why Coherent States?
- The propagator in the coherent state representation

Motivations for a path integral formulation of QM

The evolution operator $U(T) = e^{-i\hat{H}T/\hbar}$ cannot be computed explicitly,

except for very simple problems.

The infinitesimal propagator, $U(\varepsilon) = 1 - i\hat{H}\varepsilon/\hbar$ on the other hand, is

essentially the Hamiltonian operator, and can be handled more easily.

There is, therefore, a practical motivation to calculate finite time evolutions

as a composition of infinitesimal evolutions.

DIRAC: is there a Lagrangian formulation of QM?

The infinitesimal propagator:

$$\langle q'' | e^{-i\varepsilon H/\hbar} | q' \rangle = \int \langle q'' | p' \rangle \langle p' | e^{-i\frac{\varepsilon}{\hbar}\left(\frac{\hat{p}^2}{2m}+V(\hat{q})\right)} | q' \rangle \, dp'$$

$$\approx \int \langle q'' | p' \rangle \langle p' | 1 - i\frac{\varepsilon}{\hbar}\left(\frac{\hat{p}^2}{2m}+V(\hat{q})\right) | q' \rangle \, dp'$$

$$\approx \frac{1}{2\pi\hbar} \int e^{-i\frac{\varepsilon}{\hbar}\left(\frac{p'^2}{2m}+V(q')\right)+i\frac{p'}{\hbar}(q''-q')} \, dp'$$

$$\langle q''| e^{-i\varepsilon H/\hbar} |q'\rangle = \frac{1}{2\pi\hbar}\int e^{-i\frac{\varepsilon}{\hbar}\left(\frac{p'^2}{2m}+V(q')\right)+i\frac{p'}{\hbar}(q''-q')} dp'$$

$$= \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2} e^{\frac{i}{\hbar}\left[\frac{m}{2}\frac{(q''-q')^2}{\varepsilon^2} - V(q')\right]\varepsilon}$$

$$= \underbrace{\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2}}_{A} e^{\frac{i}{\hbar}S(q'',q',\varepsilon)}$$

The finite propagator

$$\langle q'' | e^{-iHT/\hbar} | q' \rangle = \int \langle q_N | e^{-iH\varepsilon/\hbar} | q_{N-1} \rangle \langle q_{N-1} | e^{-iH\varepsilon/\hbar} | q_{N-2} \rangle \cdots$$

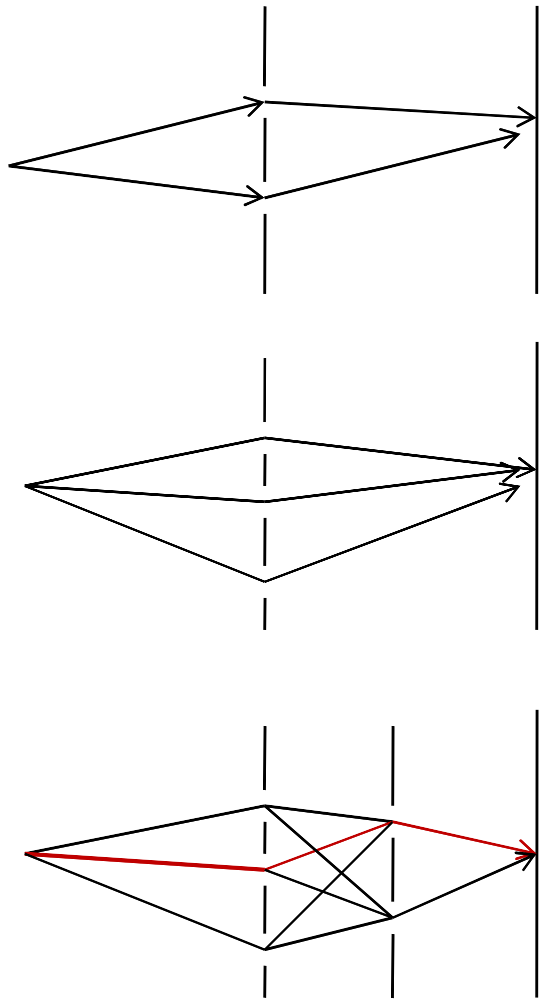
$$\langle q_1 | e^{-iH\varepsilon/\hbar} | q_0 \rangle \, dq_{N-1} dq_{N-2} \cdots dq_1$$

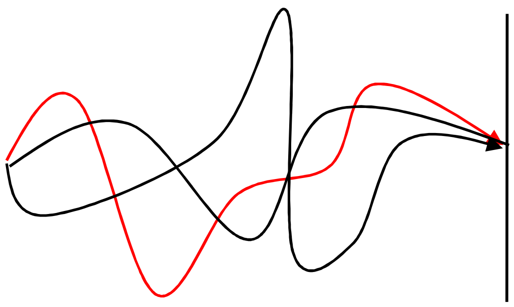
$$= \int \frac{dq_{N-1}}{A} \frac{dq_{N-2}}{A} \cdots \frac{dq_1}{A} \, e^{\frac{i}{\hbar}\sum_{n=1}^{N} S(q_n, q_{n-1}, \varepsilon)}$$

$$= \int D\left[q(t)\right] e^{iS/\hbar}$$

with $\quad q_0 \equiv q' \qquad\qquad q_N = q''$

FEYNMAM: sum over paths, generalizing the double slit experiment





$$\langle q'' | e^{-iHT/\hbar} | q' \rangle = \int D[q(t)]\, e^{iS/\hbar}$$

Quantum mechanics is written in terms of classical paths, making the connection between the two theories less mysterious.

Some Properties of the propagator

$$K(q'',q',T) \equiv \left\langle q'' \,|\, e^{-iHT/\hbar} \,|\, q' \right\rangle$$

Time evolution of wavefunctions $\quad \Psi(q'',T) = \int K(q'',q',T)\Psi(q',0)\,dq'$

Spectral decomposition $\quad K(q'',q',T) = \sum_n \varphi_n(q'')\varphi_n^*(q')\,e^{-iE_nT/\hbar}$

Trace formula $\quad \int K(q,q,T)\,dq = \sum_n e^{-iE_nT/\hbar}$

Coherent states of the Harmonic Oscillator

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{q}^2}{2} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{q} + i\frac{1}{\sqrt{2m\hbar\omega}}\hat{p} \qquad\qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{q} - i\frac{1}{\sqrt{2m\hbar\omega}}\hat{p}$$

$$\hat{a}\left|z_0\right\rangle \equiv z_0\left|z_0\right\rangle$$

$$\left|z_0\right\rangle = \hat{D}(z_0, z_0^*)\left|0\right\rangle \equiv e^{z_0\hat{a}^\dagger - z_0^*\hat{a}}\left|0\right\rangle = e^{-|z_0|^2/2}\, e^{z_0\hat{a}^\dagger}\left|0\right\rangle$$

where z_0 is a complex number.

Properties:

$$\langle z_0 | \hat{q} | z_0 \rangle = \sqrt{\frac{\hbar}{2m\omega}}(z_0 + z_0^*) \equiv q_0$$

$$\langle z_0 | \hat{p} | z_0 \rangle = -i\sqrt{\frac{m\hbar\omega}{2}}(z_0 - z_0^*) \equiv p_0$$

$$(\Delta\hat{q})^2 = \frac{\hbar}{2m\omega} \qquad (\Delta\hat{p})^2 = \frac{m\hbar\omega}{2}$$

Convenient definition:

$$z_0 = \frac{1}{\sqrt{2}}\left(\frac{q_0}{b} + i\,\frac{p_0}{c}\right)$$

$$b = \sqrt{\frac{\hbar}{m\omega}} \qquad c = \sqrt{m\hbar\omega} \qquad bc = \hbar$$

More important properties:

$$\langle q \,|\, z_0 \rangle = \pi^{-1/4}\; b^{-1/2} \exp\left[-\frac{(q-q_0)^2}{2b^2} + \frac{i p_0 q}{\hbar} \right]$$

$$\langle p \,|\, z_0 \rangle = \pi^{-1/4}\; c^{-1/2} \exp\left[-\frac{(p-p_0)^2}{2c^2} - \frac{i q_0 p}{\hbar} \right]$$

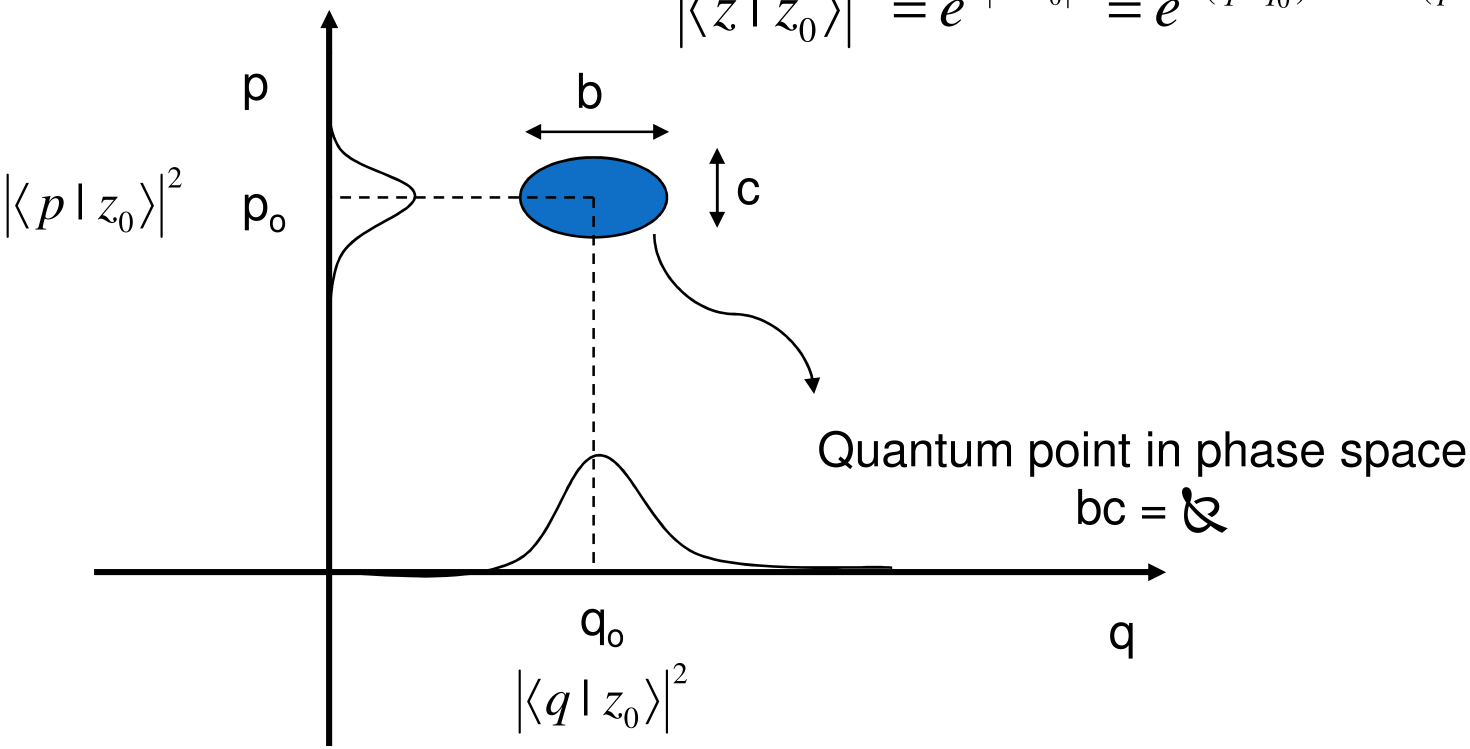
$$\langle z \,|\, z_0 \rangle = \exp\left[-\frac{1}{2}\left|z\right|^2 - \frac{1}{2}\left|z_0\right|^2 + z^* z_0 \right]$$

$$\langle n \,|\, z_0 \rangle = \frac{z_0^n}{\sqrt{n!}}\, e^{-\left|z_0\right|^2/2}$$

$$\int \left| z \right\rangle \frac{d^2 z}{\pi} \left\langle z \right| \;\equiv\; \int \left| z \right\rangle \frac{dq\, dp}{2\pi\hbar} \left\langle z \right| \;=\; 1$$

Phase space view

$$\left|\langle z \mid z_0 \rangle\right|^2 = e^{-\left|z-z_0\right|^2} = e^{-(q-q_0)^2/2b^2-(p-p_0)^2/2c^2}$$



Dynamics under H_0

$$\left|z\right\rangle(t) \equiv e^{-iH_0 t/\hbar}\left|z_0\right\rangle \equiv e^{-i\omega t/2}\left|z(t)\right\rangle$$

$$z(t) = e^{-i\omega t} z_0$$

$$q(t) = q_0 \cos(\omega t) + \frac{1}{m\omega} p_0 \sin(\omega t)$$

$$p(t) = -m\omega q_0 \sin(\omega t) + p_0 \cos(\omega t)$$

The Bargmann Representation

q-representation: $|\psi\rangle \rightarrow \psi(q) = \langle q|\psi\rangle;\qquad \hat{q}|\psi\rangle \rightarrow q\psi(q);\qquad \hat{p}|\psi\rangle \rightarrow -i\hbar\frac{\partial\psi(q)}{\partial q}$

p-representation: $|\psi\rangle \rightarrow \tilde{\psi}(p) = \langle p|\psi\rangle;\qquad \hat{p}|\psi\rangle \rightarrow p\tilde{\psi}(p);\qquad \hat{x}|\psi\rangle \rightarrow i\hbar\frac{\partial\tilde{\psi}(p)}{\partial p}$

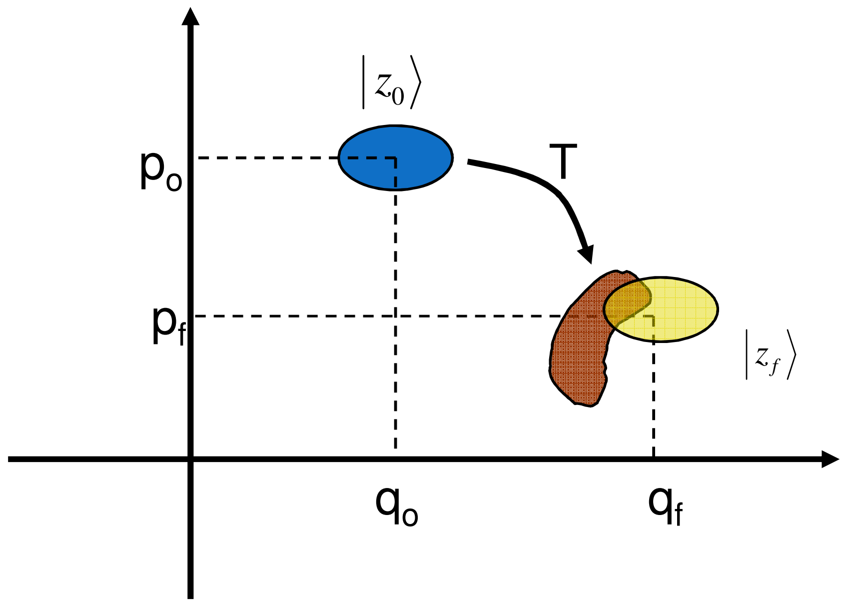
Bargmann-representation: $|z_0) = e^{z_0\hat{a}^\dagger}|0\rangle$

$$|\psi\rangle \rightarrow \psi(z^*) = (z|\psi\rangle;\qquad \hat{a}^\dagger|\psi\rangle \rightarrow z^*\psi(z^*);\qquad \hat{a}|\psi\rangle \rightarrow \frac{\partial\psi(z^*)}{\partial z^*}$$

analytic function of z^* (z does not appear)

The Coherent State Propagator

$$K(z_f, z_0, T) \equiv \langle z_f \,|\, e^{-iHT/\hbar} \,|\, z_0 \rangle = e^{-\frac{1}{2}|z_f|^2 - \frac{1}{2}|z_0|^2} \left(z_f \,|\, e^{-iHT/\hbar} \,|\, z_0 \right)$$



Path Integrals in the Coherent State Representation

$$K(z_f, z_0, T) = \langle z_N | \prod_{j=1}^{N} e^{-i\hat{H}\varepsilon/\hbar} | z_0 \rangle$$

$$= \langle z_N | \underbrace{e^{-i\varepsilon\hat{H}/\hbar} \ldots e^{-i\varepsilon\hat{H}/\hbar} \; e^{-i\varepsilon\hat{H}/\hbar} \ldots e^{-i\varepsilon\hat{H}/\hbar}}_{N} | z_0 \rangle$$

There are many ways to calculate the infinitesimal propagators

First Form: Klauder and Skagerstan (1985)

$$K(z_f, z_0, T) = \langle z_N | \prod_{j=1}^{N} e^{-i\hat{H}\varepsilon/\hbar} | z_0 \rangle$$

$$= \langle z_N | \left(1 - \frac{i\varepsilon}{\hbar}\hat{H}\right) \cdots \left(1 - \frac{i\varepsilon}{\hbar}\hat{H}\right) \underbrace{}_{\int |z_{k+1}\rangle\langle z_{k+1}| \frac{d^2 z_{k+1}}{\pi}} \left(1 - \frac{i\varepsilon}{\hbar}\hat{H}\right) \underbrace{}_{\int |z_k\rangle\langle z_k| \frac{d^2 z_k}{\pi}} \cdots \left(1 - \frac{i\varepsilon}{\hbar}\hat{H}\right) | z_0 \rangle$$

$$\left\langle z_{k+1} \middle| 1 - i\varepsilon \hat{H} / \hbar \middle| z_k \right\rangle = \left\langle z_{k+1} \middle| z_k \right\rangle \left[1 - \frac{i\varepsilon}{\hbar} \frac{\left\langle z_{k+1} \middle| \hat{H} \middle| z_k \right\rangle}{\left\langle z_{k+1} \middle| z_k \right\rangle} \right]$$

$$\equiv \left\langle z_{k+1} \middle| z_k \right\rangle \left[1 - \frac{i\varepsilon}{\hbar} H_Q^k \right]$$

$$= \left\langle z_{k+1} \middle| z_k \right\rangle e^{-i\varepsilon H_Q^k / \hbar}$$

$$= e^{z_{k+1}^* z_k - \frac{|z_k|^2}{2} - \frac{|z_{k+1}|^2}{2} - \frac{i\varepsilon H_Q^k}{\hbar}}$$

$$z_{k+1}^{*} z_{k} - \frac{|z_k|^2}{2} - \frac{|z_{k+1}|^2}{2} - \frac{i\varepsilon H_Q^k}{\hbar}$$

$$= \frac{1}{2}(z_{k+1}^{*} - z_{k}^{*}) z_k - \frac{1}{2}(z_{k+1} - z_k) z_{k+1}^{*} - \frac{i\varepsilon H_Q^k}{\hbar}$$

$$= \frac{i}{\hbar}\left[\frac{\hbar}{2i}\frac{z_{k+1}^{*} - z_k^{*}}{\varepsilon} z_k - \frac{\hbar}{2i}\frac{z_{k+1} - z_k}{\varepsilon} z_{k+1}^{*} - H_Q^k\right]\varepsilon$$

$$K_Q(z_f, z_0, T) = \int \prod_{j=1}^{N-1} \frac{d^2 z_j}{\pi} \exp\left[\sum_{j=0}^{N-1} \frac{1}{2}(z_{j+1}^* - z_j^*) z_j - \frac{1}{2} z_{j+1}^* (z_{j+1} - z_j) - \frac{i\varepsilon}{\hbar} H_Q(z_{j+1}^*, z_j) \right]$$

$$\frac{\langle z_{k-1} | \hat{H} | z_k \rangle}{\langle z_{k-1} | z_k \rangle} \quad \rightarrow \quad H_Q(z^*, z) = \langle z | \hat{H} | z \rangle$$

The Q-representation of $\hat{H}$
(normal ordering)

Since $\hat{a} | z \rangle = z | z \rangle \qquad \langle z | \hat{a}^\dagger = z^* \langle z |$

we calculate H_Q by simply writing H in *normal order*:

$$\hat{H} = \sum_{n,m} H_{nm} \hat{a}^{\dagger n} \hat{a}^m \qquad \rightarrow \qquad H_Q = \sum_{n,m} H_{nm} z^{*n} z^m$$

Example: ($\hbar$=m=1)

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{\hat{q}^2}{2} + \hat{q}^4$$

$$\hat{a} = \sqrt{\frac{1}{2}}\left(\frac{\hat{q}}{b} + i\hat{p}b\right) \qquad \hat{a}^\dagger = \sqrt{\frac{1}{2}}\left(\frac{\hat{q}}{b} - i\hat{p}b\right)$$

$$z = \frac{1}{\sqrt{2}}\left(\frac{q}{b} + ipb\right) \qquad z^* = \frac{1}{\sqrt{2}}\left(\frac{q}{b} - ipb\right)$$

$$H_Q = \frac{p^2}{2} + \frac{x^2}{2} + x^4 + 3b^2x^2 + \frac{1}{4}(b^2 + b^{-2}) + \frac{3}{4}b^4$$

Second form: Klauder and Skagerstan (1985)

$$K(z_f, z_0, T) = \langle z_N | \prod_{j=1}^{N-1} e^{-i\hat{H}\varepsilon/\hbar} | z_0 \rangle$$

$$= \langle z_N | \left(1 - \frac{i\varepsilon}{\hbar}\hat{H}\right) \ldots \left(1 - \frac{i\varepsilon}{\hbar}\hat{H}\right)\left(1 - \frac{i\varepsilon}{\hbar}\hat{H}\right) \ldots \left(1 - \frac{i\varepsilon}{\hbar}\hat{H}\right) | z_0 \rangle$$

$$\hat{H} = \int H_P(z_k{}^*, z_k) \left| z_k \right\rangle \frac{d^2 z_k}{\pi} \left\langle z_k \right|$$

$$1 - i\varepsilon \hat{H}/\hbar = \int \left[1 - i\varepsilon H_P^k/\hbar \right] \left| z_k \right\rangle \frac{d^2 z_k}{\pi} \left\langle z_k \right|$$

$$= \int e^{-i\varepsilon H_P^k/\hbar} \left| z_k \right\rangle \frac{d^2 z_k}{\pi} \left\langle z_k \right|$$

$$\left(1 - i\varepsilon \hat{H}/\hbar \right) \left(1 - i\varepsilon \hat{H}/\hbar \right)$$

$$= \int e^{-i\varepsilon H_P^k/\hbar - i\varepsilon H_P^{k+1}/\hbar} \left\langle z_{k+1} \middle| z_k \right\rangle \left| z_{k+1} \right\rangle \frac{d^2 z_k}{\pi} \left\langle z_k \right|$$

$$K_P(z_f, z_0, T) = \int \prod_{j=1}^{N-1} \frac{d^2 z_j}{\pi} \exp\left[\sum_{j=0}^{N-1} \frac{1}{2}(z^*_{j+1} - z^*_j) z_j - \frac{1}{2} z^*_{j+1}(z_{j+1} - z_j) - \frac{i\varepsilon}{\hbar} H_P(z^*_j, z_j) \right]$$

The P-representation of H
(anti-normal ordering)

Example:

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{\hat{q}^2}{2} + \hat{q}^4 \qquad H_P = \frac{p^2}{2} + \frac{x^2}{2} + x^4 - 3b^2 x^2 - \frac{1}{4}(b^2 + b^{-2}) + \frac{3}{4} b^4$$

If we take the limit $N \rightarrow \infty$ we get

$$K(z_f, z_0, T) = \begin{cases} \displaystyle\int D[z(t)]\, e^{\frac{i}{\hbar}\int\left[\frac{i\hbar}{2}(z^*\dot{z}-\dot{z}^*z)-H_Q\right]dt} \\ \\ \displaystyle\int D[z(t)]\, e^{\frac{i}{\hbar}\int\left[\frac{i\hbar}{2}(z^*\dot{z}-\dot{z}^*z)-H_P\right]dt} \end{cases}$$

It is remarkable that two apparently different path integrals, involving actions computed with different Hamiltonians, represent the same quantum object.

In fact the difference lies not only in the Hamiltonian, but also in the way it enters in the action:

$H_P(z^*_k ,z_k)$ in one case

$H_Q(z^*_{k+1},z_k)$ in the other

We will show for a simple case that the two forms are indeed identical. This emphasizes that we have to be extra careful to take the continuum limit.

FURTHER QUESTIONS:

1 – If coherent states are the *most classical* states, why these path integrals are not governed by the classical Hamiltonian?

2 – What is the transformation that takes the Hamiltonian operator into the classical Hamiltonian function?

3 – Is it possible to construct a path integral formulation where the classical H appear?

The **Weyl Symbol** of the Hamiltonian operator is defined as

$$H_W(q,p) = \int e^{ipx/\hbar} \left\langle q - \frac{x}{2} \middle| \hat{H} \middle| q + \frac{x}{2} \right\rangle dx$$

and has a number of important properties. In particular:

(a) $$\hat{H} = K(\hat{p}) + V(\hat{q}) \quad \rightarrow \quad H_W(q,p) = K(p) + V(q)$$

(b) $$\left\langle \hat{H} \right\rangle = \left\langle \Psi \middle| \hat{H} \middle| \Psi \right\rangle = \int H_w(q,p) W_\Psi(q,p)\, dq\, dp$$

where $$W_\Psi(q,p) = \int \Psi^*(q + x/2)\Psi(q - x/2)\, e^{ipx/\hbar} dx$$

is the Wigner function.

Third form:

$$K(z_f, z_0, T) = \langle z_N \mid \prod_{j=1}^{N} e^{-i\hat{H}\varepsilon/\hbar} \mid z_0 \rangle$$

$$= \langle z_N \mid \left(1 - \frac{i\varepsilon}{\hbar}\hat{H}\right) \cdots \left(1 - \frac{i\varepsilon}{\hbar}\hat{H}\right)\left(1 - \frac{i\varepsilon}{\hbar}\hat{H}\right) \cdots \left(1 - \frac{i\varepsilon}{\hbar}\hat{H}\right) \mid z_0 \rangle$$

write

$$\underbrace{\hat{H} = \int}_{\text{Weyl Hamiltonian}} H_W(w_k{}^*, w_k)\hat{D}(z, z^*)\, e^{z w_k{}^* - z^* w_k} \frac{d^2 z}{\pi} \frac{d^2 w_k}{\pi}$$

$$\hat{D}(z, z^*) = e^{z\hat{a}^\dagger - z^* \hat{a}}$$

displacement operator

$$\left\langle z_k \middle| \hat{H} \middle| z_{k-1} \right\rangle = \int H_W(w_k{}^*, w_k)\, e^{-|z|^2/2 + z z_k^* - z^* z_{k-1}} \left\langle z_k \middle| z_{k-1} \right\rangle e^{z w_k{}^* - z^* w_k}\ \frac{d^2 z}{\pi} \frac{d^2 w_k}{\pi}$$

$$\left\langle z_k \middle| e^{-i\hat{H}\varepsilon/\hbar} \middle| z_{k-1} \right\rangle = \int e^{-iH_W(w_k^*, w_k)\varepsilon/\hbar}\ e^{-|z|^2/2 + z z_k^* - z^* z_{k-1}} \left\langle z_k \middle| z_{k-1} \right\rangle e^{z w_k{}^* - z^* w_k}$$

$$\times \frac{d^2 z}{\pi} \frac{d^2 w_k}{\pi}$$

$$\left\langle z_k \left| e^{-i\hat{H}\varepsilon/\hbar} \right| z_{k-1} \right\rangle = \int e^{-iH_W(w_k^*, w_k)\varepsilon/\hbar} \; e^{-|z|^2/2 + z z_k^* - z^* z_{k-1}} \; e^{-|z_k|^2/2 - |z_{k-1}|^2/2 + z_{k-1} z_k^*}$$

$$e^{z w_k{}^* - z^* w_k} \times \frac{d^2 z}{\pi} \frac{d^2 w_k}{\pi}$$

Integrals over z and z^* are quadratic and can be performed exactly:

$$\left\langle z_k \left| e^{-i\hat{H}\varepsilon/\hbar} \right| z_{k-1} \right\rangle = 2\int e^{-iH_W(w_k^*, w_k)\varepsilon/\hbar - 2|w_k|^2 + 2 z_k^* w_k - 2 z_k w_k^* - |z_k|^2/2 - |z_{k-1}|^2/2 + z_{k-1} z_k^*} \frac{d^2 w_k}{\pi}$$

$$K_W(z_f, z_0, T) = \int \prod_{j=1}^{N} \frac{d^2 w_j}{\pi/2} \exp\left[-\frac{1}{2}|z_0|^2 - \frac{1}{2}|z_f|^2 + \psi_N + 2C_N z_f^* - 2C_N^* z_0 + z_f^* z_0 \right]$$

$$\psi_N = \sum_{k=1}^{N} \left[-\frac{i\varepsilon}{\hbar} H_W - 2|w_k|^2 \right] + 4 \sum_{k=1}^{N} \sum_{j=1}^{k} w_{k+1}^* w_{k+1-j} (-1)^{j+1}$$

$$C_N = \sum_{k=1}^{N} (-1)^{k+1} w_{N+1-k}$$

The Weyl-representation of H
(symmetric ordering)
Coelho-Aguiar J. Phys. A (2006)

Example:

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{\hat{q}^2}{2} + \hat{q}^4 \qquad H_W = \frac{p^2}{2} + \frac{x^2}{2} + x^4$$

In the limit of the continuum we may formally write

$$K_W(z_f, z_0, T) = e^{-\frac{1}{2}|z_0|^2 - \frac{1}{2}|z_f|^2 + z_f^* z_0} \int D[w, w^*] \exp\left[\psi + 2Cz_f^* - 2C^* z_0\right]$$

The special case $z_f = z_0 = z$ becomes simply

$$K_W(z, z, T) = \int D[w, w^*] \exp\left[\psi + 2Cz^* - 2C^* z\right]$$

Application: The harmonic oscillator in the three representations

Setting
$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \frac{m\omega^2}{2}x^2 = \hbar\omega\left(a^\dagger a + 1/2\right)$$

we obtain

$$H_Q(z,z^*) = \hbar\omega(zz^* + 1/2) \qquad H_W(z,z^*) = \hbar\omega zz^* \qquad H_P(z,z^*) = \hbar\omega(zz^* - 1/2)$$

All integrals in K_Q , K_P and K_W are quadratic and can be performed exactly.
We find:

$$K_Q(z_f^*, z_0, T) = \exp\left\{-i\omega T/2 + (1-i\omega\varepsilon)^N z_0 z_f^* - |z_0|^2/2 - |z_f|^2/2\right\}$$

$$K_W(z_f^*, z_0, T) = (1+i\omega\varepsilon/2)^{-N} \exp\left\{\left(\frac{1-i\omega\varepsilon}{1+i\omega\varepsilon}\right)^N z_0 z_f^* - |z_0|^2/2 - |z_f|^2/2\right\}$$

$$K_P(z_f^*, z_0, T) = (1+i\omega\varepsilon)^{-N} \exp\left\{i\omega T/2 + \frac{z_0 z_f^*}{(1+i\omega\varepsilon)^N} - |z_0|^2/2 - |z_f|^2/2\right\}$$

whereas the exact result is

$$K(z_f^*, z_0, T) = \exp\left\{-i\omega T/2 + e^{-i\omega T} z_0 z_f^* - |z_0|^2/2 - |z_f|^2/2\right\}$$

Except for the normalization and the overall multiplication phase, the non-trivial piece of the propagator is the term f=exp(-iωT) multiplying $z_i\, z_f^*$. Taking ωT=2π and N=100 we find f=1 for the exact propagator and

$$f_Q :\ 1.22+0.01i$$

$$f_W :\ 0.999998+0.002i$$

$$f_P :\ 0.82+0.007i$$

which seems to indicate a better convergence (as N→ ∞) of the Weyl form.

Relation with the Weyl Propagator

The Weyl symbol of the evolution operator

$$U(q,p,T)=\int \left\langle q-s/2 \right| e^{-i\hat{H}T/\hbar} \left| q+s/2 \right\rangle e^{ips/\hbar}\, ds$$

which is also a phase space representation of the evolution operator, can be related to the coherent state propagator as

$$U(q,p,T)=\int \left\langle q-s/2 \middle| z_0 \right\rangle \left\langle z_0 \right| e^{-i\hat{H}T/\hbar} \left| z_f \right\rangle \left\langle z_f \middle| q+s/2 \right\rangle e^{ips/\hbar} \frac{d^2 z_0}{\pi} \frac{d^2 z_f}{\pi} ds$$

Inserting the Weyl propagator, the integrals over z_0 and z_f become quadratic and can be performed exactly. These integrals result in a delta function involving the variable s, which is also performed. The final result is

$$U(q,p,T) = \int D[w,w^*]\exp\left[\psi + 2Cz^* - 2C^*z \ + 2|C|^2\right]$$

where $$z = \frac{1}{\sqrt{2}}\left(\frac{q}{b} + i\frac{pb}{\hbar}\right)$$

and $$2C_N z^* - 2C_N^* z = \sum_{k=1}^{N} \frac{2i}{\hbar}(Q_k p - P_k q)$$ is independent of b.

Above we defined the integration variables $$w_k = \frac{1}{\sqrt{2}}\left(\frac{Q_k}{b} + i\frac{P_k b}{\hbar}\right)$$

Therefore, the path integral for Weyl symbol of the evolution operator is the ‘unsmoothed’ version of the path integral for diagonal coherent state propagator.

Conversely, the path integral for diagonal coherent state propagator is identical to that for the Weyl symbol of the evolution operator, but where each path is smoothed by the Gaussian factor $|C|^2$.

Some bibliography:

The Lagrangian in Quantum Mechanics
Dirac, P. A. M. 1933 Phys. Z. Sowjet. 3, 1. Reprinted in
Schwinger, J. (ed) 1958 Quantum electrodynamics. New York: Dover. Schwinger (1958), pp. 312-320.

Space-Time Approach to Non-Relativistic Quantum Mechanics
Feynman, R. P. I948 Rev. mod. Phys. 20, 367-387.

Techniques and Applications of Path Integration
L.S. Schulman (1981) Dover

Handbook of Feynman Path Integrals *(Springer Tracts in Modern Physics)* (Berlin: Springer)
C. Grosche and F. Steiner 1998

Coherent States, Applications in Physics and Mathematical Physics
J.R. Klauder and B.S. Skagerstam 1985 (Singapore: World Scientific)

On a Hilbert Space of Analytic Functions and an Associated Integral Transform
V. Bargmann, Comm. on Pure and Appl. Math. 14, 187 (1961).

Semiclassical approximations in phase-space with coherent states
M. Baranger, M.A.M. de Aguiar, F. Keck H.J. Korsch and B. Schellaas
J. Phys. A: Math. Gen. 34 7227-7286 (2001).

Coherent State Path Integrals in the Weyl Representation
L.C. dos Santos and M.A.M. de Aguiar
J. Phys. A39 13465 (2006).

The Weyl representation in classical and quantum mechanics
A. M. Ozorio de Almeida
Phys. Rep. 295 (1998) 265.