LISTA 7 — Problema 8.1

(1) $$H = \frac{p^2}{2} + \frac{w^2 q^2}{2} + \varepsilon \left(a q + \frac{b}{2} q^2 \right)$$

(a)
$$H = \frac{p^2}{2m} + \frac{q^2}{2}(w^2 + \varepsilon b) + \varepsilon a q$$
$$= \frac{p^2}{2m} + \frac{\Omega^2}{2}\left(q + \frac{\varepsilon a}{\Omega^2}\right)^2 - \frac{\varepsilon^2 a^2}{2\Omega^2}$$

$$\begin{cases} q(t) = \left(q_0 + \frac{\varepsilon a}{\Omega^2}\right)\cos\Omega t + \frac{p_0}{\Omega}\sin\Omega t - \frac{\varepsilon a}{\Omega^2} \\ p(t) = p_0 \cos\Omega t - \Omega\left(q_0 + \frac{\varepsilon a}{\Omega^2}\right)\sin\Omega t \end{cases}$$

$$\frac{p(t)^2}{2} + \frac{\Omega^2}{2}\left(q(t) + \frac{\varepsilon a}{\Omega^2}\right)^2 - \frac{\varepsilon^2 a^2}{2\Omega^2}$$
$$= \frac{p_0^2}{2} + \left(q_0 + \frac{\varepsilon a}{\Omega^2}\right)^2 \frac{\Omega^2}{2} - \frac{\varepsilon^2 a^2}{2\Omega^2} = E$$

(b)

$$q = \sqrt{\frac{2I}{\omega}} \sin\theta$$

$$p = \sqrt{2I\omega} \cos\theta$$

$$H = \omega I + \varepsilon a \sqrt{\frac{2I}{\omega}} \sin\theta + \frac{\varepsilon b I}{\omega} \sin^2\theta$$

$$= H_0 + \varepsilon H_1$$

$$H_1 = a\sqrt{\frac{2I}{\omega}} \sin\theta + \frac{\varepsilon b I}{\omega}\left[\frac{1-\cos 2\theta}{2}\right]$$

$$= \frac{\varepsilon b I}{2\omega} + a\sqrt{\frac{2I}{\omega}} \sin\theta - \frac{\varepsilon b I}{2\omega}\cos 2\theta$$

$$\langle H_1 \rangle = \frac{\varepsilon b I}{2\omega}$$

$$\tilde{H}_1 = H_1 - \langle H_1 \rangle = a\sqrt{\frac{2I}{\omega}} \sin\theta - \frac{\varepsilon b I}{2\omega}\cos 2\theta$$

$$S_1 = -\frac{1}{\omega}\int \tilde{H}_1 \, d\theta$$

$$= +\frac{a}{\omega}\sqrt{\frac{2I}{\omega}} \cos\theta + \frac{\varepsilon b I}{4\omega^2}\sin 2\theta$$

Transformação canônica $(I, \theta) \to (J, \varphi)$

$$S(J,\theta) = J\cdot\theta + \varepsilon\frac{a}{\omega}\sqrt{\frac{2J}{\omega}}\cos\theta + \varepsilon\frac{bJ}{4\omega^2}\sin 2\theta$$

$$I = \frac{\partial S}{\partial \theta} = J + \varepsilon\frac{\partial S_1}{\partial \theta} \approx J + \varepsilon\frac{\partial S_1}{\partial \varphi}$$

$$\varphi = \frac{\partial S}{\partial J} = \theta + \frac{\varepsilon \partial S_1}{\partial J} \Rightarrow \theta \approx \varphi - \varepsilon\frac{\partial S_1}{\partial J}$$

$$H(J) = H_0(J) + \varepsilon \langle H_1 \rangle$$

$$\boxed{H(J) = \omega J + \frac{\varepsilon b J}{2\omega}}$$

$$J = J_0$$

$$\varphi = \varphi_0 + \left(\omega + \frac{\varepsilon b}{2\omega}\right) t \equiv \varphi_0 + \tilde{\omega} t$$

$$q = \sqrt{\frac{2I}{\omega}} \sin\theta = \sqrt{\frac{2}{\omega}\left(J + \varepsilon \frac{\partial S_1}{\partial \varphi}\right)} \sin\left(\varphi - \varepsilon \frac{\partial S_1}{\partial J}\right)$$

$$= \sqrt{\frac{2J}{\omega}} \sin\varphi \left[1 + \frac{\varepsilon}{2J} \frac{\partial S_1}{\partial \varphi}\right] \left[1 - \varepsilon \frac{\partial S_1}{\partial J} \operatorname{cotg} \varphi\right]$$

$$\approx \sqrt{\frac{2J}{\omega}} \sin\varphi \left\{1 + \varepsilon \left[\frac{1}{2J} \frac{\partial S_1}{\partial \varphi} - \frac{\partial S_1}{\partial J} \operatorname{cotg} \varphi\right]\right\}$$

$$p = \sqrt{2I\omega} \cos\theta = \sqrt{2\omega\left(J + \varepsilon \frac{\partial S_1}{\partial \varphi}\right)} \cos\left(\varphi - \varepsilon \frac{\partial S_1}{\partial J}\right)$$

$$= \sqrt{2\omega J} \cos\varphi \left[1 + \frac{\varepsilon}{2J} \frac{\partial S_1}{\partial \varphi}\right] \left[1 + \varepsilon \frac{\partial S_1}{\partial J} \operatorname{tg} \varphi\right]$$

$$\approx \sqrt{2\omega J} \cos\varphi \left\{1 + \varepsilon \left[\frac{1}{2J} \frac{\partial S_1}{\partial \varphi} + \frac{\partial S_1}{\partial J} \operatorname{tg} \varphi\right]\right\}$$

$$\frac{\partial S_1}{\partial \varphi} = \frac{1}{w}\left[-a\sqrt{\frac{2J}{w}}\sin\varphi + \frac{bJ}{2w}\cos 2\varphi\right]$$

$$\frac{\partial S_1}{\partial J} = \frac{1}{w}\left[\frac{a}{\sqrt{2Jw}}\cos\varphi + \frac{b}{4w}\sin 2\varphi\right]$$

$$\frac{1}{2J}\frac{\partial S_1}{\partial \varphi} - \frac{\partial S_1}{\partial J}\operatorname{cotg}\varphi =$$

$$= -\frac{a}{2wJ}\sqrt{\frac{2J}{w}}\sin\varphi + \frac{b}{4w^2}\cos 2\varphi - \frac{a}{w\sqrt{2Jw}}\frac{\cos^2\varphi}{\sin\varphi} - \frac{b}{4w^2}\frac{\cos\varphi}{\sin\varphi}\,2\sin\varphi\cos\varphi$$

$$= \frac{-a}{w\sqrt{2Jw}}\left[\frac{\sin^2\varphi}{\sin\varphi} + \frac{\cos^2\varphi}{\sin\varphi}\right] + \frac{b}{4w^2}\left[\cos^2\varphi - \sin^2\varphi - 2\cos^2\varphi\right]$$

$$= \frac{-a}{w\sqrt{2Jw}}\frac{1}{\sin\varphi} - \frac{b}{4w^2}$$

$$q(t) = \sqrt{\frac{2J}{w}}\sin\varphi\left\{1 - \frac{\varepsilon}{w\sqrt{2Jw}}\frac{a}{\sin\varphi} - \frac{b\varepsilon}{4w^2}\right\}$$

$$\boxed{q(t) = \sqrt{\frac{2J}{w}}\left[1 - \frac{b\varepsilon}{4w^2}\right]\sin\varphi - \frac{\varepsilon a}{w^2}}$$

$$\frac{1}{2I}\frac{\partial S_1}{\partial \varphi} + \frac{\partial S_1}{\partial J}\, tg\,\varphi =$$

$$-\frac{a}{\cancel{2wJ}}\sqrt{\frac{2J}{w}}\sin\varphi + \frac{b}{4w^2}\cos 2\varphi + \frac{a}{\cancel{w\sqrt{2Jw}}}\sin\varphi + \frac{b}{4w^2}\sin 2\varphi\, tg\,\varphi =$$

$$\frac{b}{4w^2}\left[\cos^2\varphi - \sin^2\varphi + 2\sin\varphi\cos\varphi\,\frac{\sin\varphi}{\cos\varphi}\right] = \frac{b}{4w^2}$$

$$\boxed{p = \sqrt{2Jw}\left[1+\frac{\varepsilon b}{4w^2}\right]\cos\varphi}$$

CONDIÇÕES INICIAIS $\qquad \varphi = \varphi_0 + \tilde{w}\, t$

$$q(t) = \sqrt{\frac{2J}{w}}\left(1-\frac{\varepsilon b}{4w^2}\right)\sin\varphi_0\cos\tilde{w}t + \sqrt{\frac{2J}{w}}\left(1-\frac{\varepsilon b}{4w^2}\right)\cos\varphi_0\sin\tilde{w}t - \frac{\varepsilon a}{w^2}$$

$$p(t) = \sqrt{2Jw}\left(1+\frac{\varepsilon b}{4w^2}\right)\cos\varphi_0\cos\tilde{w}t - \sqrt{2Jw}\left(1+\frac{\varepsilon b}{4w^2}\right)\sin\varphi_0\sin\tilde{w}t$$

$$\tilde{q}_0 = \sqrt{\frac{2J}{w}}\left(1-\frac{\varepsilon b}{4w^2}\right)\sin\varphi_0 - \frac{\varepsilon a}{w} \Rightarrow \sqrt{\frac{2J}{w}}\left(1-\frac{\varepsilon b}{4w^2}\right)\sin\varphi_0 \equiv \tilde{q}_0 + \frac{\varepsilon a}{w}$$

$$\tilde{p}_0 = \sqrt{2Jw}\left(1+\frac{\varepsilon b}{4w^2}\right)\cos\varphi_0$$

$$q(t) = \left(\tilde{q}_0 + \frac{\varepsilon a}{w}\right)\cos\tilde{w}t + \frac{1}{w}\frac{(1-\varepsilon b/4w^2)}{(1+\varepsilon b/4w^2)}\tilde{p}_0\sin\tilde{w}t - \frac{\varepsilon a}{w^2}$$

$$\frac{1}{w}\frac{1-\varepsilon b/4w^2}{1+\varepsilon b/4w^2} \simeq \frac{1}{w(1+\varepsilon b/2w^2)} = \frac{1}{\tilde{w}}$$

$$\boxed{q(t) = \left(\tilde{q}_0 + \frac{\varepsilon a}{\omega}\right)\cos\tilde{\omega} t + \frac{\tilde{p}_0}{\tilde{\omega}}\sin\tilde{\omega} t - \frac{\varepsilon a}{\omega^2}}$$

$$p(t) = \tilde{p}_0 \cos\tilde{\omega} t - \omega\left(\frac{1+\varepsilon b/4\omega^2}{1-\varepsilon b/4\omega^2}\right)\left(\tilde{q}_0 + \frac{\varepsilon a}{\omega}\right)\sin\tilde{\omega} t$$

$$\boxed{p(t) = \tilde{p}_0 \cos\tilde{\omega} t - \tilde{\omega}\left(\tilde{q}_0 + \frac{\varepsilon a}{\omega}\right)\sin\tilde{\omega} t}$$

Comparando com o cálculo exato:

$$\Omega = \omega\sqrt{1+\frac{\varepsilon b}{\omega^2}} \approx \omega\left(1+\frac{\varepsilon b}{2\omega^2}\right) = \omega + \frac{\varepsilon b}{2\omega} = \tilde{\omega}$$