Steepest Descent for Pedestrians

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1. Preliminaries:

A function of the complex variable z is a rule that associates to each z another complex number w, so that

w = f(z).

Setting z = x + i y and w = u + i v we can write

$$f(z) = u(x, y) + iv(x, y)$$

where u and v are real functions of the real variables x and y.

The function f(z) has a derivative at z_0 if

$$\lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}$$

exists and converges to the same value independent on the way δz goes to zero. In particular, we may set

(a) $\delta z = \delta x$ to get

$$\frac{df}{dz} = \lim_{\delta x \to 0} \frac{u(x_0 + \delta x, y_0) + iv(x_0 + \delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\delta x}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

or, (b) $\delta z = i\,\delta y$:

$$\frac{df}{dz} = \lim_{\delta y \to 0} \frac{u(x_0, y_0 + \delta y) + iv(x_0, y_0 + \delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i\,\delta y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating (a) and (b) we get the Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The function f is analytic in an open set D if it has derivatives at all points of D.

2. Critical points of f(z):

Suppose there is a point z_0 such that $\frac{\partial f}{\partial z}(z_0) = 0$. In the vicinity of z_0 we can

expand the real part of f(z) as:

$$u(x,y) = u(x_0,y_0) + \frac{1}{2}\underbrace{\frac{\partial^2 u}{\partial x^2}}_{b}\delta x^2 + \underbrace{\frac{\partial^2 u}{\partial x \partial y}}_{-a}\delta x \delta y + \frac{1}{2}\frac{\partial^2 u}{\partial y^2}\delta y^2$$

$$\equiv u_0 + \frac{1}{2} b\,\delta x^2 - a\,\delta x \delta y - \frac{1}{2} b\,\delta y^2$$

$$= u_0 + \frac{1}{2}\begin{pmatrix}\delta x & \delta y\end{pmatrix}\underbrace{\begin{pmatrix} b & -a \\ -a & -b \end{pmatrix}}_{A}\begin{pmatrix}\delta x \\ \delta y\end{pmatrix}$$

The first b and –a in the quadratic form are definitions. The fact that the last term is –b comes out of the Cauchy-Riemann conditions.

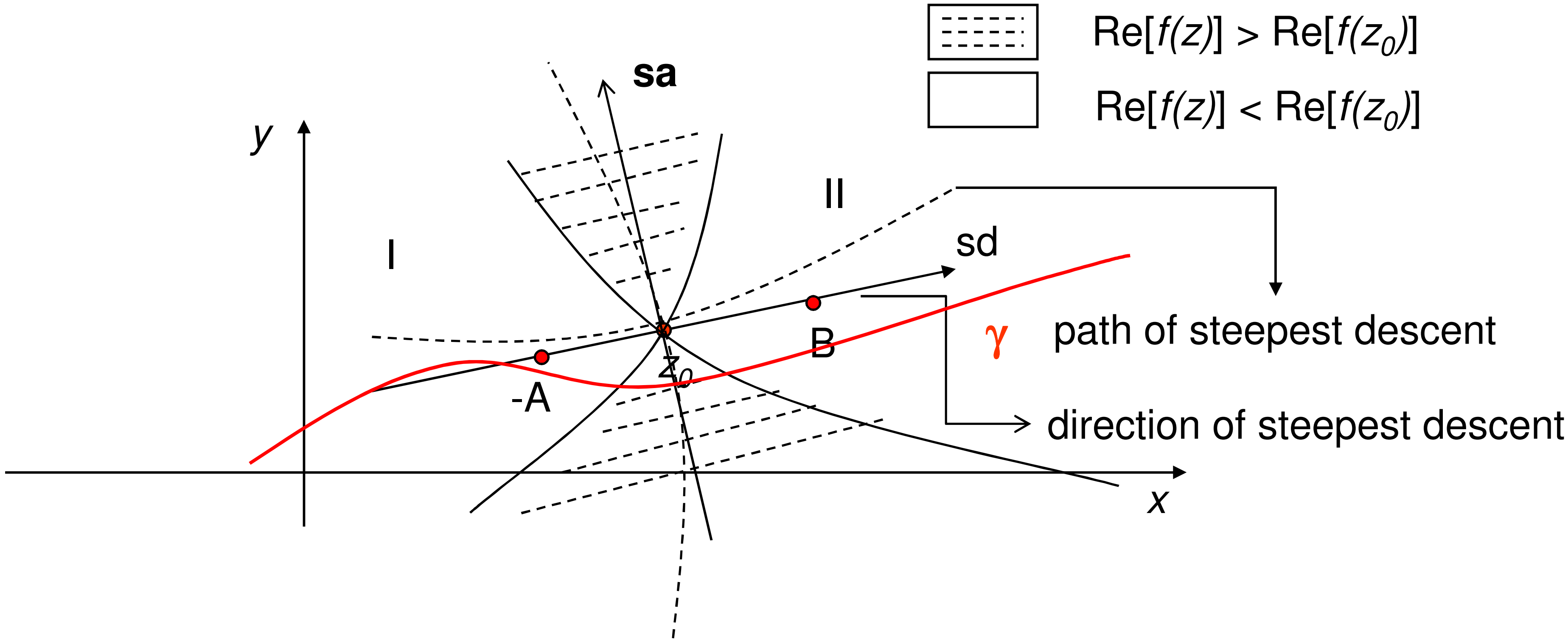
The eigenvalues of the matriz A are

$$\lambda_{\pm} = \pm\sqrt{a^2 + b^2}$$

Therefore, z_0 is a saddle point. The corresponding (non-normalized) eigenvectors

$$t_{\pm} = \begin{pmatrix} a \\ b \mp \sqrt{a^2 + b^2} \end{pmatrix}$$

are orthogonal to each other and define the directions of steepest descent **sd** (t_-) and steepest ascent **sa** (t_+) from $z_{0.}$ This is illustrated in the next page:



The line **sd** passing through z_0 is given explictly by

$$y = y_0 + \frac{b+\sqrt{a^2+b^2}}{a}(x-x_0) \equiv y_0 + \mu(x-x_0)$$

Another important feature of the steepest descent line **sd** is that the imaginary part of f(z) remains constant along it. This can be seen explicitly by expanding v(x,y) and calculating it on the **sd** line:

$$v(x,y) = v(x_0, y_0) + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}\delta x^2 + \frac{\partial^2 v}{\partial x \partial y}\delta x \delta y + \frac{1}{2}\frac{\partial^2 v}{\partial y^2}\delta y^2$$

$$\equiv v_0 + \frac{1}{2}a\,\delta x^2 + b\,\delta x \delta y - \frac{1}{2}a\,\delta y^2$$

The coefficients a and b are the same as defined before. Once again we have to use the Cauchy-Riemann conditions to figure this out. Setting

$$\delta y = \mu\,\delta x$$

where μ is defined in the previous page, one can easily check that the quadratic terms cancel out and $v(x,y) = v_0$.

Note: the dotted white curves in the figure will be called the steepest descent and steepest ascent *paths*. These paths are defined as the curves through z_0 where the imaginary part of f(z) is kept constant. The **sd** and **sa** lines are the tangents to these curves at z_0. We can easily prove that the gradient of u(x,y) is indeed always tangent to these paths. So far we have only proved this for the lines in the vicinity of $z_{0.}$ The paths are given implicitly by

$$v(x, y) = v(x_0, y_0)$$

If (x+dx,y+dy) is also on the path, then

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = 0 \qquad \rightarrow \qquad \vec{\nabla} v \bullet d\vec{x} = 0$$

i.e., the gradient of v(x,y) is perpendicular to the curve. Using Cauchy-Riemann we can re-write this as

$$-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy = 0 \qquad \rightarrow \qquad \vec{\nabla} u \times d\vec{x} = 0$$

i.e., the gradient of u(x,y) is parallel to the curves. These are, therefore, the curves of steepest descent or ascent.

3. The Steepest Descent Approximation

Consider the integral below where *f(z)* is analytic in the domain *D* containing the path γ , and suppose t is a large parameter.

$$I(t) = \int_{\gamma} e^{tf(z)} dz$$

Since f is analytic we can deform the path smoothly without changing the value of the integral. In many cases of interest γ is just the real line.

If the path can be deformed so as to pass through z_0 via the steepest descent path, and if t is sufficiently large, then the integrand is going to be significantly different from zero only in its immediate neighborhood, that we can approximate by the steepest descent direction. There, the imaginary part of f(z) remains constant and the real part behaves like a Gaussian function, allowing us to perform the integral.

Under these conditions we expand f(z) for z on the steepest descent line **sd**. We get

$$f(z) = f(z_0) + \frac{1}{2} b\, \delta x^2 - a\, \delta x \delta y - \frac{1}{2} b\, \delta y^2 \,|_{\delta y = \mu \delta x}$$

Notice that only the real part of f is contributing to the quadratic terms, since those coming from the imaginary part cancel on the **sd** line**.** Remember that

$$\mu = \frac{b + \sqrt{a^2 + b^2}}{a} \equiv \frac{b + \lambda}{a}$$

After replacing $\delta y = \mu\, \delta x$ we get

$$f(z) = f(z_0) - \underbrace{\left(\frac{a^2 + b^2}{a}\right)\left(\frac{b + \sqrt{a^2 + b^2}}{a}\right)}_{> 0} \delta x^2 = -\frac{\lambda^2 \mu}{a} \delta x^2$$

Also

$$dz = dx + i dy = (1 + i\mu)\, dx = \sqrt{1+\mu^2}\, e^{i\theta} dx$$

and

$$I(t) \approx \sqrt{1+\mu^2}\, e^{i\theta} e^{t f(z_0)} \int_{-A}^{+B} e^{-t\lambda^2 \mu \delta x^2 / a} d(\delta x)$$

$$= \sqrt{\frac{a}{t\lambda^2 \mu}} \sqrt{1+\mu^2}\, e^{i\theta} e^{t f(z_0)} \int_{-A\sqrt{t\lambda^2\mu/a} \to -\infty}^{+B\sqrt{t\lambda^2\mu/a} \to \infty} e^{-\xi^2} d\xi$$

$$= \sqrt{\frac{\pi a (1+\mu^2)}{t\lambda^2 \mu}}\, e^{t f(z_0) + i\theta}$$

A and B are two points on the **sd** line (see previous figure). When t is large we can take the limits of integration to infinity. θ is the angle **sd** makes with the x-axis.

The pre-factor can be simplified: first we note that

$$\mu = (b+\lambda)/a \qquad\qquad 1+\mu^2 = 2\lambda(b+\lambda)/a^2$$

Therefore

$$\frac{\pi a(1+\mu^2)}{t\lambda^2\mu} = \frac{\pi a 2\lambda(b+\lambda)}{t\lambda^2(b+\lambda)a} = \frac{2\pi}{t\lambda}$$

and

$$I(t) \approx \sqrt{\frac{2\pi}{t\lambda}}\; e^{t\,f(z_0)+i\theta}$$

The last step is to relate the pre-factor and the angle θ, with the second derivative of f(z) at z_0. From page 3 and the Cauchy-Riemann conditions we have

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$\frac{d^2 f}{dz^2} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right) - i\frac{\partial^2 u}{\partial x \partial y}$$

Applying at z_0 and using our definition of coefficients a and b we find

$$\frac{d^2 f}{dz^2}(z_0) = b + ia$$

Therefore, $$\lambda = \sqrt{a^2 + b^2} = \left|\frac{d^2 f}{dz^2}(z_0)\right| \equiv \left|f''(z_0)\right|$$

We now show that the phase θ can be written as $(\pi-\alpha)/2$, where $\alpha=\arg[f''(z_0)]$, i.e., $\tan(\alpha)=a/b$. We start from

$$\tan(\pi/2-\alpha/2)=\frac{1}{\tan(\alpha/2)}\equiv\frac{1}{\zeta}$$

Also

$$\tan(\alpha)=\frac{2\tan(\alpha/2)}{1-\tan(\alpha/2)^2}\equiv\frac{2\zeta}{1-\zeta^2}\quad\rightarrow\quad\zeta^2+\frac{2b}{a}\zeta-1=0$$

where $\zeta=\tan(\alpha/2)$ and $\tan(\alpha)=a/b$. The solutions are

$$\zeta_\pm=-\frac{b}{a}\pm\frac{\sqrt{a^2+b^2}}{a}\qquad\text{and}\qquad\zeta_\pm^{-1}=\frac{b}{a}\pm\frac{\sqrt{a^2+b^2}}{a}$$

Taking the plus solution we find $\zeta^{-1}=\mu=\tan(\theta)$. Therefore

$$\tan(\pi/2-\alpha/2)=\tan(\theta)$$

The final result is

$$I(t) = \int_\gamma e^{tf(z)} dz \approx \sqrt{\frac{2\pi}{t\left|f''(z_0)\right|}}\; e^{tf(z_0)+i(\pi-\alpha)/2}$$

Notice that the phase α can be put back into the square root, removing the modulus of the second derivative. The extra phase π can also be eliminated if we change f(z) into –f(z). Therefore we can also write the simpler result

$$I(t) = \int_\gamma e^{-tf(z)} dz \approx \sqrt{\frac{2\pi}{t\, f''(z_0)}}\; e^{-tf(z_0)}$$

where it is understood that the phase of f ” has to be taken out of the square root like indicated in the expression of top.

Important final comments:

1. There might be more than one saddle point. If the path γ can be deformed so as to pass through these points, their contributions should be added to the integral.

2. If two or more saddle points are too close together, better approximations are needed.

3. Finding the saddle points is not enough: one need to check if they contribute, i.e., if the path can be deformed so as to pass through them.