Complemento para a questão 1

Quando tratamos uma partícula com spin podemos pensar em conjunto completo de operadores na forma

$$A,\ L^2,\ L_z,\ S^2,\ S_z$$

Os estados de base são

$$|n\ell m\varepsilon\rangle = |n\ell m\rangle \otimes |\varepsilon\rangle$$

e $\quad \langle r\theta\varphi|n\ell m\rangle = R_{n\ell}(r)\, Y_{\ell m}(\theta,\varphi)$.

Assim

$$|\psi\rangle = \sum_{n\ell m\varepsilon} a^{\varepsilon}_{n\ell m}\, |n\ell m\rangle \otimes |\varepsilon\rangle$$

Projetando em $\langle r\theta\varphi|$ obtemos uma função mista de r,θ,φ que ainda envolve os kets $|\varepsilon\rangle$ do spin:

$$||\psi(r,\theta,\varphi)\rangle \equiv \sum_{\substack{n\ell m\\ \varepsilon}} a^{\varepsilon}_{n\ell m}\, R_{n\ell}(r)\, Y_{\ell m}(\theta,\varphi)\, |\varepsilon\rangle$$

$$= \begin{pmatrix} \psi^{+}(r,\theta,\varphi) \\ \psi^{-}(r,\theta,\varphi) \end{pmatrix}$$

onde $\quad \psi^{\varepsilon}(r,\theta,\varphi) = \sum_{n\ell m} a^{\varepsilon}_{n\ell m}\, R_{n\ell}(r)\, Y_{\ell m}(\theta,\varphi)$

A probabilidade de se medir L^2 e L_z e obter l e m, por exemplo, é

$$P_{lm} = \sum_{n,\varepsilon} \left| a^{\varepsilon}_{nlm} \right|^2 .$$

No problema 1 a função de onda é apresentada de outra forma. Para escrever $\|\Psi\rangle$ dessa maneira fazemos

$$\|\Psi(r,\theta,\varphi)\rangle = \sum_{l,m,\varepsilon} \underbrace{\left(\sum_{n} a^{\varepsilon}_{nlm} R_{nl}(r) \right)}_{C^{\varepsilon}_{lm}(r)} Y_{lm}(\theta,\varphi) |\varepsilon\rangle$$

$$= \sum_{l,m,\varepsilon} C^{\varepsilon}_{lm}(r)\, Y_{lm}(\theta,\varphi)\, |\varepsilon\rangle$$

$$= \underbrace{\sum_{l,m} C^{+}_{lm}(r)\, Y_{lm}(\theta,\varphi) |+\rangle}_{\Psi^+} + \underbrace{\sum_{lm} C^{-}_{lm}(r)\, Y_{lm} |-\rangle}_{\Psi^-}$$

Agora

$$P_{lm} = \sum_{\varepsilon} \int_0^{\infty} \left| C^{\varepsilon}_{lm}(r) \right|^2 r^2\, dr$$

Prova:

Como $$c^{\varepsilon}_{\ell m}(r) = \sum_n a^{\varepsilon}_{n\ell m} R_{n\ell}(r)$$

$$\int_0^\infty |c^{\varepsilon}_{\ell m}(r)|^2 r^2 dr = \sum_{n,n'} \int a^{\varepsilon *}_{n'\ell m} R^{*}_{n'\ell}(r)\, a^{\varepsilon}_{n\ell m} R_{n\ell}\, r^2 dr$$

$$= \sum_{nn'} a^{\varepsilon *}_{n'\ell m}\, a^{\varepsilon}_{n\ell m} \underbrace{\int_0^\infty R^{*}_{n'\ell}(r) R_{n\ell}(r)\, r^2 dr}_{\delta_{nn'}}$$

$$= \sum_n |a^{\varepsilon}_{n\ell m}|^2$$

e

$$\sum_{\varepsilon} \int_0^\infty |c^{\varepsilon}_{\ell m}(r)|^2 r^2 dr = \sum_{\varepsilon, n} |a^{\varepsilon}_{n\ell m}|^2 = P_{\ell m}$$

Na questão 1 os únicos coeficientes NÃO nulos são

$$c^+_{00} = R(r) \qquad c^+_{10} = \frac{1}{\sqrt{3}} R(r)$$

$$c^-_{11} = \frac{R(r)}{\sqrt{3}} \qquad c^-_{10} = -\frac{R(r)}{\sqrt{3}}$$

então

$$P(L_z=0) = \int \left[|c^+_{00}|^2 + |c^+_{10}|^2 + |c^-_{10}|^2 \right] r^2 dr$$

$$= \left(1 + \frac{1}{3} + \frac{1}{3}\right) \int_0^\infty |R(r)|^2 r^2 dr = \frac{5}{3} \cdot \frac{1}{2} = \frac{5}{6}$$

$$P(L_z=\hbar) = \int |c^-_{11}|^2 r^2 dr = \frac{1}{3} \int_0^\infty |R(r)|^2 r^2 dr = \frac{1}{6}$$

$$P(S_z=\hbar/2) = \int \left[|c^+_{00}|^2 + |c^+_{10}|^2 \right] r^2 dr = \left(1 + \frac{1}{3}\right) \cdot \frac{1}{2} = \frac{2}{3}$$

$$P(S_z=-\hbar/2) = \int \left[|c^-_{11}|^2 + |c^-_{10}|^2 \right] r^2 dr = \left(\frac{1}{3} + \frac{1}{3}\right) \cdot \frac{1}{2} = \frac{1}{3}$$