26. Why must an eigenfunction be well behaved in order to be acceptable in the Schroedinger theory?
27. Explain in two or three sentences how the quantization of energy is related to the well-behaved character of acceptable eigenfunctions.
28. Why is ψ necessarily an oscillatory function if $V(x) < E$?
29. Why does ψ tend to go to infinity if $V(x) > E$?
30. Is it ever possible for an allowed value of the total energy E of a system to be less than the minimum value of its potential energy $V(x)$? Give a qualitative argument, along the lines of the arguments in Section 5-7, to justify your answer.
31. We have seen several examples of the general result that the lowest allowed value of the total energy E, for a particle bound in a potential $V(x)$, lies above the minimum value of $V(x)$. Use the uncertainty principle in a qualitative argument to explain why this must be so.
32. If a particle is not bound in a potential, its total energy is not quantized. Does this mean the potential has *no* effect on the behavior of the particle? What effect would you expect it to have?

PROBLEMS

1. If the wave functions $\Psi_1(x,t)$, $\Psi_2(x,t)$, and $\Psi_3(x,t)$ are three solutions to the Schroedinger equation for a particular potential $V(x,t)$, show that the arbitrary linear combination $\Psi(x,t) = c_1\Psi_1(x,t) + c_2\Psi_2(x,t) + c_3\Psi_3(x,t)$ is also a solution to that equation.
2. At a certain instant of time, the dependence of a wave function on position is as shown in Figure 5-20. (a) If a measurement that could locate the associated particle in an element dx of the x axis were made at that instant, where would it most likely be found? (b) Where would it least likely be found? (c) Are the chances better that it would be found at *any* positive value of x, or are they better that it would be found at *any* negative value of x? (d) Make a rough sketch of the potential $V(x)$ which gives rise to the wave function. (e) To which allowed energy does the wave function correspond?
3. (a) Determine the frequency ν of the time-dependent part of the wave function, quoted in Example 5-3, for the lowest energy state of a simple harmonic oscillator. (b) Use this value of ν, and the de Broglie-Einstein relation $E = h\nu$, to evaluate the total energy E of the oscillator. (c) Use this value of E to show that the limits of the classical motion of the oscillator, found in Example 5-6, can be written as $x = \pm\hbar^{1/2}/(Cm)^{1/4}$.
4. By evaluating the classical normalization integral in Example 5-6, determine the value of the constant B^2 which satisfies the requirement that the total probability of finding the particle in the classical oscillator somewhere between its limits of motion must equal one.
5. Use the results of Examples 5-5, 5-6, and 5-7 to evaluate the probability of finding a particle, in the lowest energy state of a quantum mechanical simple harmonic oscillator,

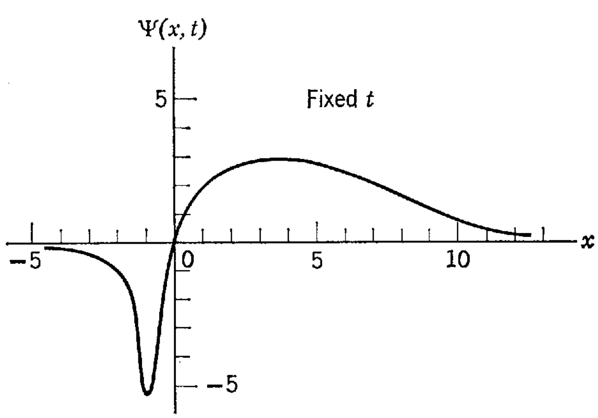

Figure 5-20 The space dependence of a wave function considered in Problem 2, evaluated at a certain instant of time.

within the limits of the classical motion. (Hint: (i) The classical limits of motion are expressed in a convenient form in the statement of Problem 3c. (ii) The definite integral that will be obtained can be expressed as a normal probability integral, or an error function. It can then be evaluated immediately by consulting mathematical handbooks which tabulate these quantities. Or, the integral can easily be evaluated by expanding the exponential as an inifinite series *before* integrating, and then integrating the first few terms in the series. Alternatively, the definite integral can be evaluated by plotting the integrand on graph paper, and counting squares to find the area enclosed between the integrand, the axis, and the limits.)

6. At sufficiently low temperature, an atom of a vibrating diatomic molecule is a simple harmonic oscillator in its lowest energy state because it is bound to the other atom by a linear restoring force. (The restoring force is linear, at least approximately, because the molecular vibrations are very small.) The force constant C for a typical molecule has a value of about $C \sim 10^3$ nt/m. The mass of the atom is about $m \sim 10^{-26}$ kg. (a) Use these numbers to evaluate the limits of the classical motion from the formula quoted in Problem 3c. (b) Compare the distance between these limits to the dimensions of a typical diatomic molecule, and comment on what this comparison implies concerning the behavior of such a molecule at very low temperatures.

7. Use the particle in a box wave function verified in Example 5-9, with the value of A determined in Example 5-10, to calculate the probability that the particle associated with the wave function would be found in a measurement within a distance of $a/3$ from the right-hand end of the box of length a. The particle is in its lowest energy state. (b) Compare with the probability that would be predicted classically from a very simple calculation related to the one in Example 5-6.

8. Use the results of Example 5-9 to estimate the total energy of a neutron of mass about 10^{-27} kg which is assumed to move freely through a nucleus of linear dimensions of about 10^{-14} m, but which is strictly confined to the nucleus. Express the estimate in MeV. It will be close to the actual energy of a neutron in the lowest energy state of a typical nucleus.

9. (a) Following the procedure of Example 5-9, verify that the wave function

$$\Psi(x,t) = \begin{cases} A \sin \dfrac{2\pi x}{a} e^{-iEt/\hbar} & -a/2 < x < +a/2 \\ 0 & x < -a/2 \text{ or } x > +a/2 \end{cases}$$

is a solution to the Schroedinger equation in the region $-a/2 < x < +a/2$ for a particle which moves freely through the region but which is strictly confined to it. (b) Also determine the value of the total energy E of the particle in this first excited state of the system, and compare with the total energy of the ground state found in Example 5-9. (c) Plot the space dependence of this wave function. Compare with the ground state wave function of Figure 5-7, and give a qualitative argument relating the difference in the two wave functions to the difference in the total energies of the two states.

10. (a) Normalize the wave function of Problem 9, by adjusting the value of the multiplicative constant A so that the total probability of finding the associated particle somewhere in the region of length a equals one. (b) Compare with the value of A obtained in Example 5-10 by normalizing the ground state wave function. Discuss the comparison.

11. Calculate the expectation value of x, and the expectation value of x^2, for the particle associated with the wave function of Problem 10.

12. Calculate the expectation value of p, and the expectation value of p^2, for the particle associated with the wave function of Problem 10.

13. (a) Use quantities calculated in the preceding two problems to calculate the product of the uncertainties in position and momentum of the particle in the first excited state of the system being considered. (b) Compare with the uncertainty product when the particle is in the lowest energy state of the system, obtained in Example 5-10. Explain why the uncertainty products differ.

14. (a) Calculate the expectation values of the kinetic energy and the potential energy for a particle in the lowest energy state of a simple harmonic oscillator, using the wave function of Example 5-7. (b) Compare with the time-averaged kinetic and potential energies for a classical simple harmonic oscillator of the same total energy.

15. In calculating the expectation value of the product of position times momentum, an ambiguity arises because it is not apparent which of the two expressions

$$\overline{xp} = \int_{-\infty}^{\infty} \Psi^* x\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi\,dx$$

$$\overline{px} = \int_{-\infty}^{\infty} \Psi^* \left(-i\hbar\frac{\partial}{\partial x}\right)x\Psi\,dx$$

should be used. (In the first expression $\partial/\partial x$ operates on Ψ; in the second it operates on $x\Psi$.) (a) Show that neither is acceptable because both violate the obvious requirement that $\overline{xp}$ should be real since it is measurable. (b) Then show that the expression

$$\overline{xp} = \int_{-\infty}^{\infty} \Psi^* \left[\frac{x\left(-i\hbar\frac{\partial}{\partial x}\right) + \left(-i\hbar\frac{\partial}{\partial x}\right)x}{2}\right]\Psi\,dx$$

is acceptable because it does satisfy this requirement. (Hint: (i) A quantity is real if it equals its own complex conjugate. (ii) Try integrating by parts. (iii) In any realistic case the wave function will always vanish at $x = \pm\infty$.)

16. Show by direct substitution into the Schroedinger equation that the wave function

$$\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$$

satisfies that equation if the eigenfunction $\psi(x)$ satisfies the time-independent Schroedinger equation for a potential $V(x)$.

17. (a) Write the classical wave equation for a string of density per unit length which varies with x. (b) Then separate it into two ordinary differential equations, and show that the equation in x is very analogous to the time-independent Schroedinger equation.

18. By using an extension of the procedure leading to (5-31), obtain the Schroedinger equation for a particle of mass m moving in three dimensions (described by rectangular coordinates x, y, z).

19. (a) Separate the Schroedinger equation of Problem 18, for a time-independent potential, into a time-independent Schroedinger equation and an equation for the time dependence of the wave function. (b) Compare to the corresponding one-dimensional equations, (5-37) and (5-38), and explain the similarities and the differences.

20. (a) Separate the time-independent Schroedinger equation of Problem 19 into three time-independent Schroedinger equations, one in each of the coordinates. (b) Compare them with (5-37). (c) Explain clearly what must be assumed about the form of the potential energy in order to make the separation possible, and what the physical significance of this assumption is. (d) Give an example of a system that would have such a potential.

21. Starting with the relativisitic expression for the energy, formulate a Schroedinger equation for photons, and solve it by separation of variables, assuming $V = 0$.

22. Consider a particle moving under the influence of the potential $V(x) = C|x|$, where C is a constant, which is illustrated in Figure 5-21. (a) Use qualitative arguments, very similar to those of Example 5-12, to make a sketch of the first eigenfunction and of the tenth eigenfunction for the system. (b) Sketch both of the corresponding probability density functions. (c) Then use the classical mechanics to calculate, in the manner of Example 5-6, the probability density functions predicted by that theory. (d) Plot the classical probability density functions with the quantum mechanical probability density functions, and discuss briefly their comparison.

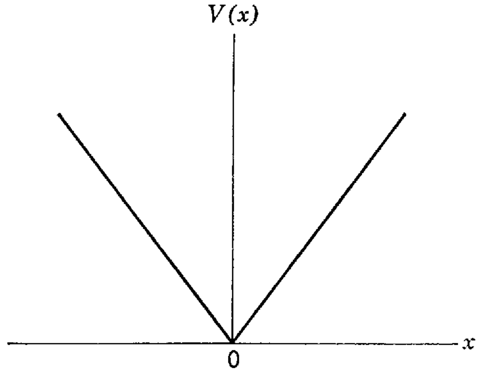

Figure 5-21 A potential function considered in Problem 22.

23. Consider a particle moving in the potential $V(x)$ plotted in Figure 5-22. For the following ranges of the total energy E, state whether there are any allowed values of E and if so, whether they are discretely separated or continuously distributed. (a) $E < V_0$, (b) $V_0 < E < V_1$, (c) $V_1 < E < V_2$, (d) $V_2 < E < V_3$, (e) $V_3 < E$.

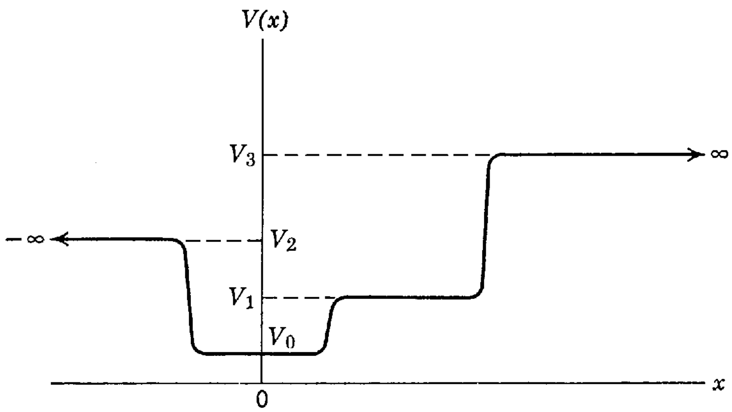

Figure 5-22 A potential function considered in Problem 23.

24. Consider a particle moving in the potential $V(x)$ illustrated in Figure 5-23, that has a rectangular region of depth V_0, and width a, in which the particle can be bound. These parameters are related to the mass m of the particle in such a way that the lowest allowed energy E_1 is found at an energy about $V_0/4$ above the "bottom." Use qualitative arguments to sketch the approximate shape of the corresponding eigenfunction $\psi_1(x)$.

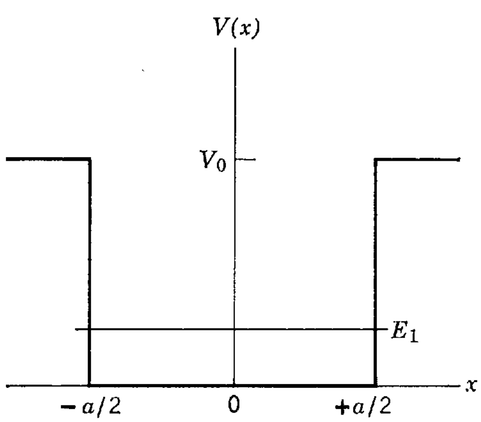

Figure 5-23 A potential function considered in Problem 24.

25. Suppose the bottom of the potential function of Problem 24 is changed by adding a bump in the center of height about $V_0/10$ and width $a/4$. That is, suppose the potential now

looks like the illustration of Figure 5-24. Consider qualitatively what will happen to the curvature of the eigenfunction in the region of the bump, and how this will, in turn, affect the problem of obtaining an acceptable behavior of the eigenfunction in the region outside the binding region. From these considerations predict, qualitatively, what the bump will do to the value of the lowest allowed energy E_1.

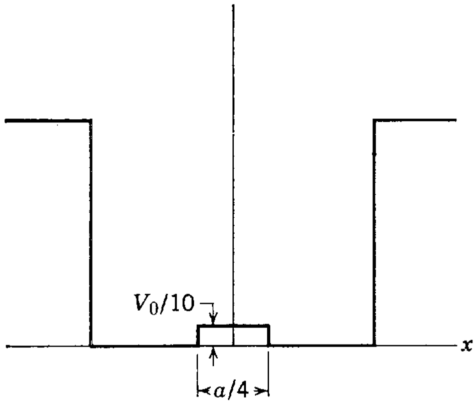

Figure 5-24 A rectangular bump added to the bottom of the potential of Figure 5-23; for Problem 25.

26. Because the bump in Problem 25 is small, a good approximation to the lowest allowed energy of the particle in the presence of the bump can be obtained by taking it as the sum of the energy in the absence of the bump plus the expectation value of the extra potential energy represented by the bump, taking the Ψ corresponding to no bump to calculate the expectation value. Using this point of view, predict whether a bump of the same "size," but located at the edge of the bottom as in Figure 5-25, would have a larger, smaller, or equal effect on the lowest allowed energy of the particle, compared to the effect of a centered bump. (Hint: Make a rough sketch of the product of $\Psi^*\Psi$ and the potential energy function that describes the centered bump. Then consider qualitatively the effect of moving the bump to the edge on the integral of this product.)

27. By substitution into the time-independent Schroedinger equation for the potential illustrated in Figure 5-23, show that in the region to the right of the binding region the eigenfunction has the mathematical form

$$\psi(x) = Ae^{-[\sqrt{2m(V_0 - E)}/\hbar]x} \qquad x > +a/2$$

28. Using the probability density corresponding to the eigenfunction of Problem 27, write an expression to estimate the distance D outside the binding region of the potential within which there would be an appreciable probability of finding the particle. (Hint: Take D to extend to the point at which $\Psi^*\Psi$ is smaller than its value at the edge of the binding region by a factor of e^{-1}. This e^{-1} criterion is similar to one often used in the study of electrical circuits.)

29. The potential illustrated in Figure 5-23 gives a good description of the forces acting on an electron moving through a block of metal. The energy difference $V_0 - E$, for the highest energy electron, is the work function for the metal. Typically, $V_0 - E \simeq 5$ eV. (a) Use this value to estimate the distance D of Problem 28. (b) Comment on the results of the estimate.

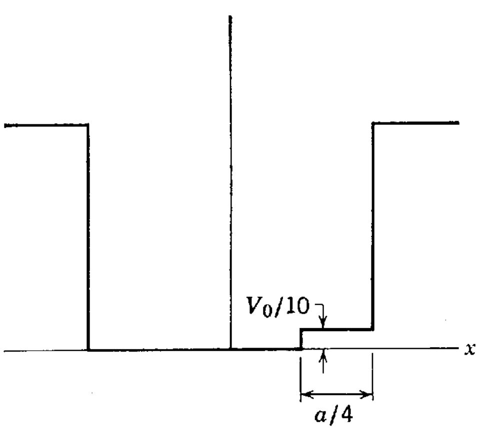

Figure 5-25 The same rectangular bump as in Figure 5-24, but moved to the edge of the potential; for Problem 26.

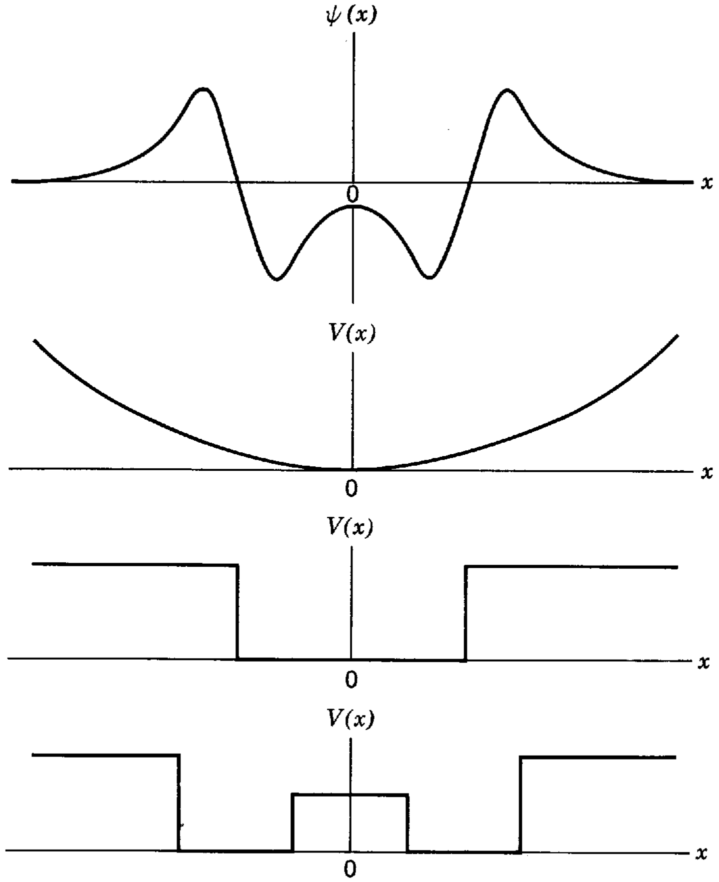

Figure 5-26 An eigenfunction (*top* curve) and three possible forms (*bottom* curves) of the potential energy function considered in Problem 30.

30. Consider the eigenfunction illustrated in the top part of Figure 5-26. (a) Which of the three potentials illustrated in the bottom part of the figure could lead to such an eigenfunction? Give qualitative arguments to justify your answer. (b) The eigenfunction shown is not the one corresponding to the lowest allowed energy for the potential. Sketch the form of the eigenfunction which does correspond to the lowest allowed energy E_1. (c) Indicate on another sketch the range of energies where you would expect discretely separated allowed energy states, and the range of energies where you would expect the allowed energies to be continuously distributed. (d) Sketch the form of the eigenfunction which corresponds to the second allowed energy E_2. (e) To which energy level does the eigenfunction presented in Figure 5-26 correspond?

31. Estimate the lowest energy level for a one-dimensional infinite square well of width a containing a cosine bump. That is, the potential V is

$$V = V_0 \cos(\pi x/a) \qquad -a/2 < x < +a/2$$
$$V = \text{infinity} \qquad x < -a/2 \text{ or } x > +a/2$$

where $V_0 \ll \pi^2\hbar^2/2ma^2$.

32. Using the first two normalized wave functions $\Psi_1(x,t)$ and $\Psi_2(x,t)$ for a particle moving freely in a region of length a, but strictly confined to that region, construct the linear combination

$$\Psi(x,t) = c_1\Psi_1(x,t) + c_2\Psi_2(x,t)$$

Then derive a relation involving the adjustable constants c_1 and c_2 which, when satisfied, will ensure that $\Psi(x,t)$ is also normalized. The normalized $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are obtained in Example 5-10 and Problem 10.

33. (a) Using the normalized "mixed" wave function of Problem 32, calculate the expectation value of the total energy E of the particle in terms of the energies E_1 and E_2 of the two states and of the values c_1 and c_2 of the mixing parameters. (b) Interpret carefully the meaning of your result.

34. If the particle described by the wave function of Problem 32 is a proton moving in a nucleus, it will give rise to a charge distribution which oscillates in time at the same frequency as the oscillations of its probability density. (a) Evaluate this frequency for values of E_1 and E_2 corresponding to a proton mass of 10^{-27} kg and a nuclear dimension of 10^{-14} m. (b) Also evaluate the frequency and energy of the photon that would be emitted by this oscillating charge distribution as the proton drops from the excited state to the ground state. (c) In what region of the electromagnetic spectrum is such a photon?