PROPAGAÇÃO DO PACOTE GAUSSIANO

$$\psi(x,0) = \frac{1}{\pi^{1/4} d^{1/2}} e^{\frac{ip_0 x}{\hbar} - \frac{x^2}{2d^2}}$$

$$K(x,x',t) = \sqrt{\frac{m}{2\pi i\hbar t}}\, e^{\frac{im(x-x')^2}{2t\hbar}}$$

$$\psi(x,t) = \int_{-\infty}^{+\infty} dx'\, K(x,x',t)\, \psi(x',0)$$

$$= \frac{1}{\pi^{1/4}} \sqrt{\frac{m}{2\pi i\hbar t d}} \int_{-\infty}^{+\infty} e^{\frac{im(x-x')^2}{2t\hbar} + \frac{ip_0 x'}{\hbar} - \frac{x'^2}{2d^2}}\, dx'$$

O expoente fica

$$-\frac{x'^2}{2}\underbrace{\left(\frac{1}{d^2} - \frac{im}{\hbar t}\right)}_{\alpha} + \frac{ix'}{\hbar}\underbrace{\left(p_0 - \frac{mx}{t}\right)}_{\beta} + \frac{imx^2}{2t\hbar}$$

e vamos usar que

$$\int_{-\infty}^{+\infty} dx'\, e^{-\frac{x'^2\alpha}{2} + i\frac{\beta x'}{\hbar}} = \sqrt{\frac{2\pi}{\alpha}}\, e^{-\beta^2/2\alpha\hbar^2}.$$

$$\psi(x,t) = \frac{1}{\pi^{\frac{1}{4}}} \sqrt{\frac{m}{2\pi i\hbar t d}} \sqrt{\frac{2\pi}{\alpha}}\, e^{\frac{imx^2}{2t\hbar} - \frac{\beta^2}{2\alpha\hbar^2}}$$

Simplificando:

$$\beta = p_0 - \frac{mx}{t} = -\frac{m}{t}\left(x - p_0 t/m\right)$$

$$\alpha = \frac{1}{d^2} - \frac{im}{\hbar t} = \frac{\hbar t - imd^2}{\hbar t d^2}$$

1) termo na raiz quadrada

$$\frac{m}{i\hbar t d \alpha} = \frac{m}{i\hbar t d}\,\frac{\hbar t d^2}{\hbar t - imd^2} = \frac{md}{md^2 + i\hbar t} = \frac{1}{d\left(1 + \frac{i\hbar t}{md^2}\right)}$$

2) expoente

$$\frac{\beta^2}{2\hbar^2\alpha} = \frac{m^2}{t^2}\,\frac{(x - p_0 t/m)^2}{2\hbar^2}\,\frac{\hbar t d^2}{\hbar t - imd^2} = \frac{m(x - p_0 t/m)^2}{2\hbar t}\,\frac{md^2}{\hbar t - imd^2}$$

$$= \frac{im}{2\hbar t}\,\frac{(x - p_0 t/m)^2}{\left(1 + \frac{i\hbar t}{md^2}\right)} \quad ; \quad \boxed{z \equiv 1 + \frac{i\hbar t}{md^2}}$$

$$\frac{\beta^2}{2\hbar^2\alpha} = \frac{im}{2\hbar t}\,\frac{(x - p_0 t/m)^2}{z} = \frac{im}{2\hbar t}\,\frac{(x - p_0 t/m)^2}{|z|^2}\,z^*$$

Assim,

$$\psi(x,t) = \frac{1}{\pi^{1/4}}\,\frac{1}{\sqrt{dz}}\,\exp\left\{\frac{imx^2}{2\hbar t} - \frac{im}{2\hbar t}\,\frac{(x - p_0 t/m)^2}{|z|^2}\,z^*\right\}$$

e

$$|\psi(x,t)|^2 = \frac{1}{\sqrt{\pi}\, d\,|z|} \exp\left\{ -\frac{m}{2\hbar t} \cdot \frac{(x - p_0 t/m)^2}{|z|^2} (iz^* - iz) \right\}$$

$$iz^* - iz = \left(i + \frac{\hbar t}{md^2}\right) - \left(i - \frac{\hbar t}{md^2}\right) = \frac{2\hbar t}{md^2}$$

Definindo $\Delta(t) = d\,|z| = d\sqrt{1 + \frac{\hbar^2 t^2}{m^2 d^2}}$

$$\boxed{|\psi(x,t)|^2 = \frac{1}{\sqrt{\pi}\,\Delta(t)} \exp\left\{ -\frac{(x - p_0 t/m)^2}{\Delta(t)^2} \right\}}$$

$$\int_{-\infty}^{+\infty} e^{-ax^2 + ibx}\, dx = \sqrt{\frac{\pi}{a}}\, e^{-b^2/4a}$$