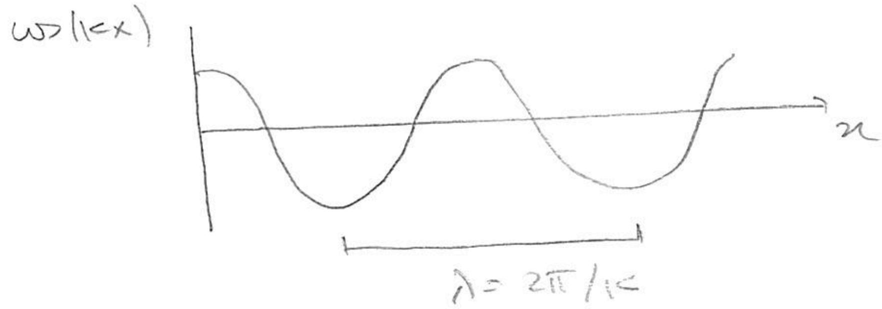
PACOTES DE ONDA

I) Partícula livre

$$\Psi_k(x,t) = e^{i(kx-\omega t)} = \text{onda que propaga p/ direita}$$

$$\Psi_k(x,t) = e^{-i(kx+\omega t)} = \text{onda que propaga p/ esquerda}$$

$$\omega(k) = \frac{\hbar k^2}{2m} \quad , \quad p \equiv \hbar k \quad , \quad E \equiv \frac{\hbar^2 k^2}{2m}$$



PACOTE COM 3 ondas planas

$$\Psi(x,t) = e^{i(k_0 x - \omega(k_0) t)} + \frac{1}{2}\left[e^{i\left(k_0+\frac{\Delta k}{2}\right)x - i\omega\left(k_0+\frac{\Delta k}{2}\right)t} + e^{i\left(k_0-\frac{\Delta k}{2}\right)x - i\omega\left(k_0-\frac{\Delta k}{2}\right)t} \right]$$

$$\omega\left(k_0 \pm \frac{\Delta k}{2}\right) \simeq \omega(k_0) \pm \frac{d\omega}{dk}(k_0)\,\frac{\Delta k}{2} = \omega(k_0) \pm v_0 \frac{\Delta k}{2}$$

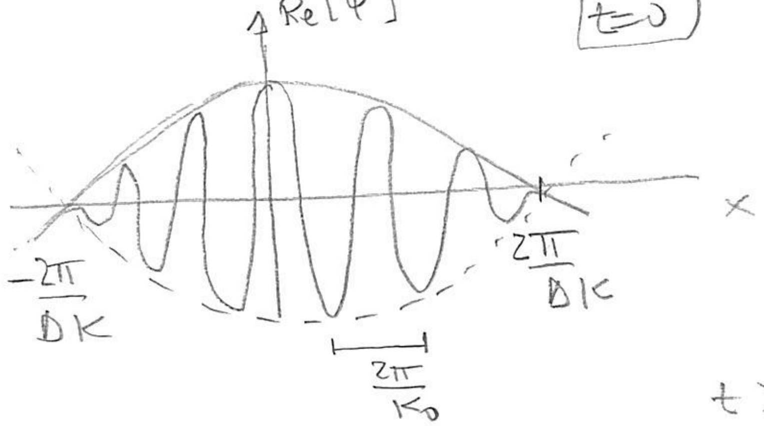
onde $\frac{d\omega}{dk}(k_0) = \frac{\hbar k_0}{m} \equiv \frac{p_0}{m} \equiv v_0$.

$$\Psi(x,t) \simeq e^{i(k_0 x - w(k_0)t)} \left[1 + \frac{e^{i(x - v_0 t)\frac{\Delta k}{2}} + e^{-i(x - v_0 t)\frac{\Delta k}{2}}}{2} \right]$$

$$= e^{i(k_0 x - w_0 t)} \underbrace{\left\{ 1 + \cos\left[(x - v_0 t) \frac{\Delta k}{2} \right] \right\}}_{f(x,t)}$$

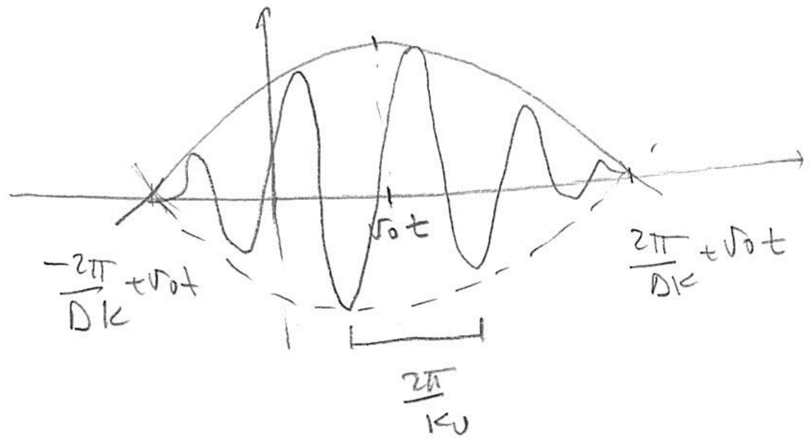
$$\text{Re}\left[\Psi(x,t)\right] = \cos(k_0 x - w_0 t)\, f(x,t)$$

$\boxed{t=0}$



$$f(x,0) = 1 + \cos(x \Delta K/2)$$

$t > 0$



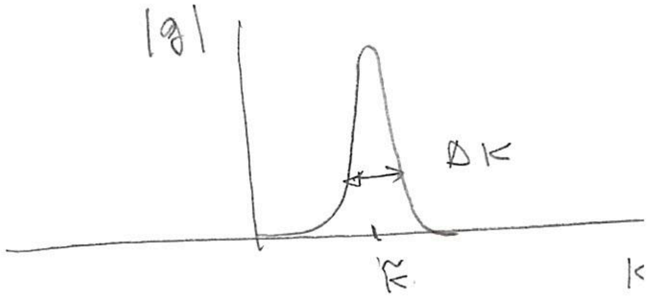
O envelope $f(x,t)$ se desloca com velocidade $v_0 = \frac{\hbar k}{m}$

A onda dentro do envelope se desloca com $v_f = \frac{v_0}{2}$

PACOTE GERAL

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int g(k)\, e^{i(kx - \omega(k)\, t)}$$

$$g(k) = |g(k)|\, e^{i\alpha(k)}$$



$$kx = \tilde{k}x + (k - \tilde{k})x$$

$$\omega(k) \approx \omega(\tilde{k}) + \frac{d\omega}{dk}(\tilde{k})(k-\tilde{k}) \equiv \tilde{\omega} + \tilde{v}(k-\tilde{k})$$

$$\alpha(k) \approx \alpha(\tilde{k}) + \frac{d\alpha}{dk}(\tilde{k})(k-\tilde{k}) \equiv \tilde{\alpha} - x_0 (k-\tilde{k})$$

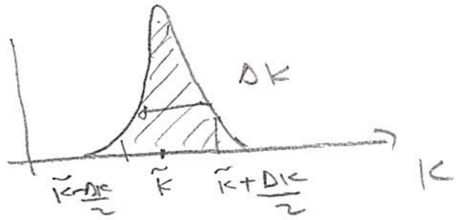
onde

$$\tilde{\omega} = \omega(\tilde{k})$$
$$\tilde{v} = \frac{d\omega}{dk}(\tilde{k}) = \frac{\hbar \tilde{k}}{m}$$
$$\tilde{\alpha} = \alpha(\tilde{k})$$
$$x_0 = -\frac{d\alpha}{dk}(\tilde{k})$$

$$\psi(x,t) \approx \frac{1}{\sqrt{2\pi}} e^{i(\tilde{k}x + \tilde{\alpha} - \tilde{\omega} t)} \underbrace{\int dk\, |g(k)|\, e^{i(k-\tilde{k})[x - x_0 - \tilde{v} t]}}_{I}$$

$$\mathrm{Re}(I) = \int dk\, |g(k)| \cos[(k-\tilde{k})(x - x_0 - \tilde{v} t)]$$

se $x = x_0 + \tilde{v} t$ $\qquad Re(I) = \int dk \, |g(k)|$

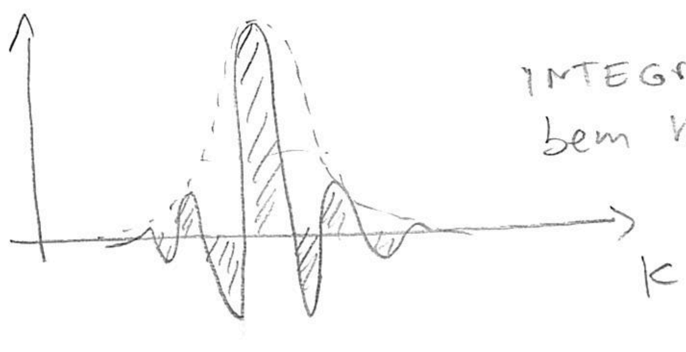
$=$ 

A integral atinge seu valor máximo.

se $x = x_0 + \tilde{v} t + \frac{4\pi}{\Delta k}$, $\cos\left[(k-\tilde{k})(x - x_0 - \tilde{v} t)\right]$

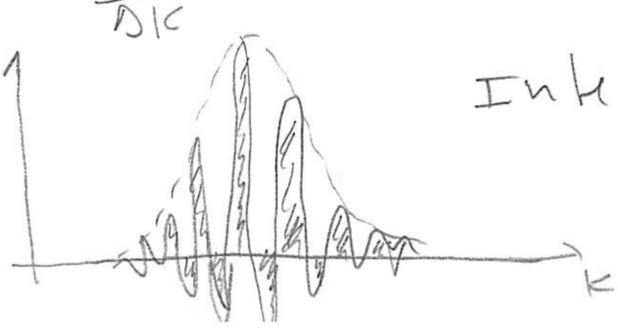
$$= \cos\left[(k-\tilde{k})\frac{4\pi}{\Delta k}\right] = \begin{cases} 1 & \text{se} \quad k = \tilde{k} \\ -1 & \text{se} \quad k = \tilde{k} + \frac{\Delta k}{4} \\ +1 & \text{se} \quad k = \tilde{k} + \frac{\Delta k}{2} \end{cases}$$

$$Re(I) = \int dk \, |g(k)| \cos(k-\tilde{k})\frac{4\pi}{\Delta k}$$

$=$ 

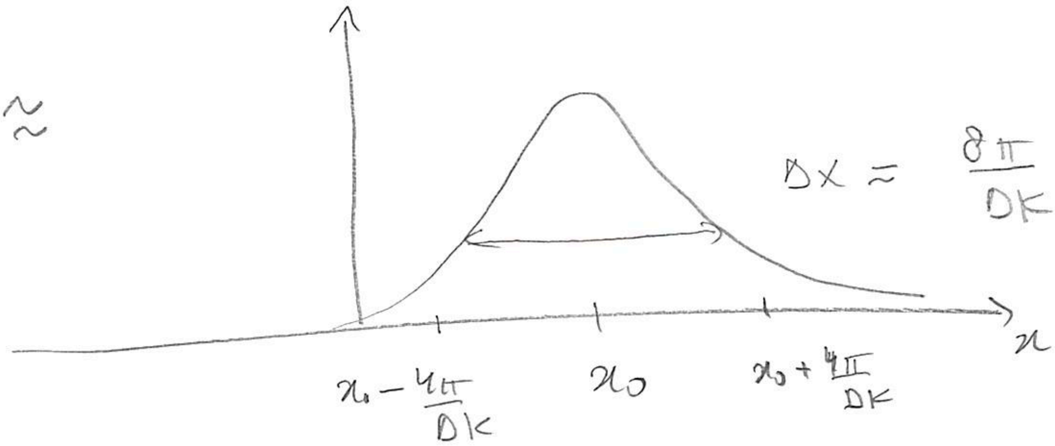
INTEGRAL JÁ é bem menor

se $x \gg x_0 + \tilde{v} t + \frac{4\pi}{\Delta k}$

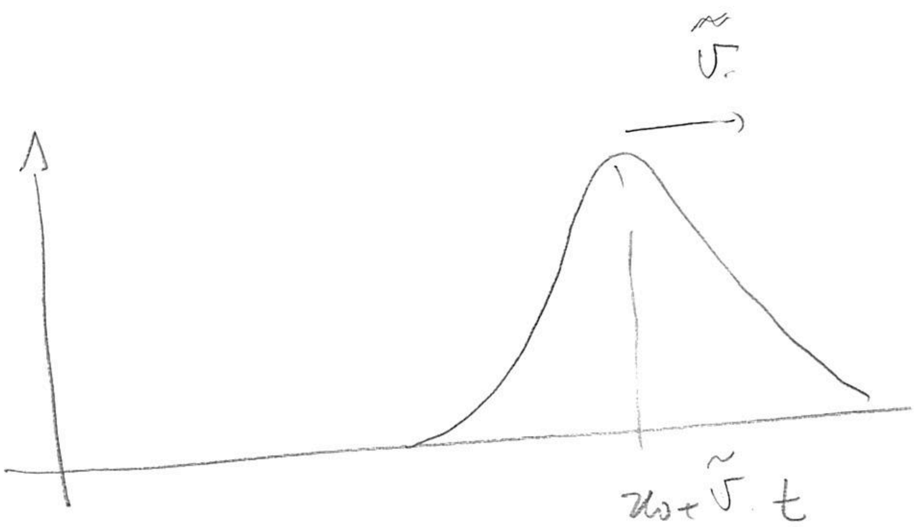


Integral $\simeq 0$

$\Rightarrow \psi(x,0) \approx$



$\psi(x,t) \approx$



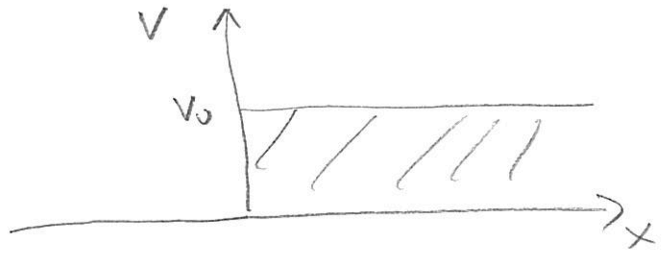
– PACOTE SE DESLOCA com velocidade $\tilde{v} = \frac{\hbar \tilde{K}}{m}$

– largura do pacote

$$\Delta x \simeq \frac{8\pi}{\Delta K} \quad ; \quad \hbar \Delta K = \Delta p$$

$$\Delta x \, \Delta p \simeq \frac{8\pi}{\hbar}$$

II - Potencial Degrau , $E < V_0$



$$\Psi(x,t) = \varphi(x)\, e^{-iEt/\hbar}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2} + V(x)\,\varphi(x) = E\,\varphi(x)$$

$$\varphi(x) = \begin{cases} A e^{ikx} + B e^{-ikx} , & x<0 \qquad k=\sqrt{2mE}/\hbar \\ C e^{-Rx} , & x>0 \qquad R=\sqrt{2m(V_0-E)}/\hbar \end{cases}$$

continuidade de $\varphi(x)$: $A + B = C$

continuidade de $\frac{d\varphi}{dx}$: $ik(A-B) = -RC$

$$\varphi(x) = A \begin{cases} e^{ikx} + \frac{ik+R}{ik-R}\, e^{-ikx} & x<0 \\ \frac{2ik}{ik-R}\, e^{-Rx} & x>0 \end{cases}$$

$$\frac{ik+R}{ik-R} = \frac{k-iR}{k+iR} = \frac{re^{-i\theta}}{re^{i\theta}} = e^{-2i\theta} \; ; \quad \boxed{tg\,\theta = \frac{R}{k}}$$

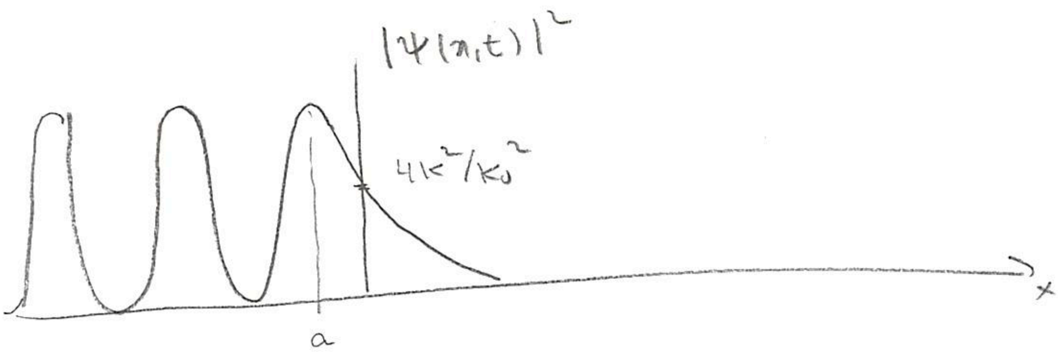
$$|\Psi(x,t)|^2 = |A|^2 \begin{cases} |e^{ikx} + e^{-ikx-2i\theta}|^2 = 2 + 2\cos(2kx + 2\theta) \\ \dfrac{4k^2}{k^2+\kappa^2} = \dfrac{4k^2}{k_0^2} \end{cases} ; \quad \boxed{k_0 = \frac{\sqrt{2mV_0}}{\hbar}}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = \frac{2}{1+\tan^2\theta} - 1$$

$$= \frac{2}{1+\kappa^2/k^2} - 1 = \frac{2k^2 - k^2 - \kappa^2}{k^2+\kappa^2} = \frac{k^2-\kappa^2}{k^2+\kappa^2} = \frac{k^2-\kappa^2}{k_0^2}$$

$$= \frac{2k^2 - k_0^2}{k_0^2} = \frac{2k^2}{k_0^2} - 1$$

$$\Rightarrow \quad 2 + 2\cos(2\theta) = 4k^2/k_0^2$$

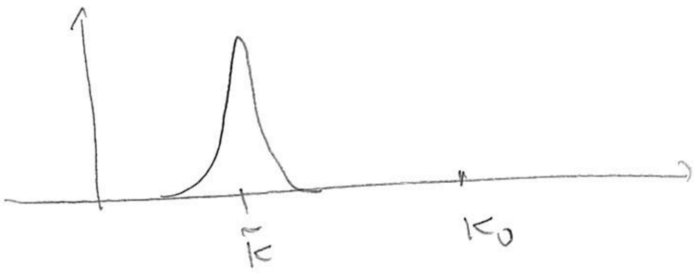


$\boxed{\text{PADRÃO ESTACIONÁRIO}}$

$$2ka + 2\theta = 0 \qquad a = -\theta/k$$

PACOTE DE ONDAS

Escolhemos $g(k) = |g(k)|\, e^{i\alpha(k)}$ com



$\tilde{k} < k_0$

e $\alpha'(\tilde{k}) \equiv -x_0 > 0 \Rightarrow$ $x_0 = -|x_0| < 0$ pacote à esquerda do degrau

Para $x < 0$ e $t = 0$

$$\psi(x,0) = \frac{1}{\sqrt{2\pi}} \int dk\, |g(k)| \left[e^{ikx + i\alpha} + e^{-ikx - 2i\theta + i\alpha} \right]$$

$$\simeq \frac{1}{\sqrt{2\pi}} \left\{ e^{i(\tilde{k}x + \tilde{\alpha})} \int |g(k)|\, e^{i(x - x_0)(k - \tilde{k})}\, dk \; + \right.$$

$$\left. e^{-i\tilde{k}x - 2i\tilde{\theta} + i\tilde{\alpha}} \int |g(k)|\, e^{-i(x + 2\theta' + x_0)(k - \tilde{k})}\, dk \right\}$$

CÁLCULO DE $\theta'(\tilde{k})$

$$\operatorname{tg}\theta = \frac{\sqrt{k_0^2 - k^2}}{k} \to \frac{1}{\cos^2\theta}\frac{d\theta}{dk} = \frac{-k}{k\sqrt{k_0^2 - k^2}} - \frac{\sqrt{k_0^2 - k^2}}{k^2} = \frac{-k_0^2/k^2}{\sqrt{k_0^2 - k^2}}$$

$$\cos^2\theta = \frac{1}{1 + \operatorname{tg}^2\theta} = \frac{1}{1 + \frac{k_0^2 - k^2}{k^2}} = \frac{k^2}{k_0^2} \Rightarrow \frac{d\theta}{dk}(\tilde{k}) = -\frac{1}{\sqrt{k_0^2 - \tilde{k}^2}} < 0$$

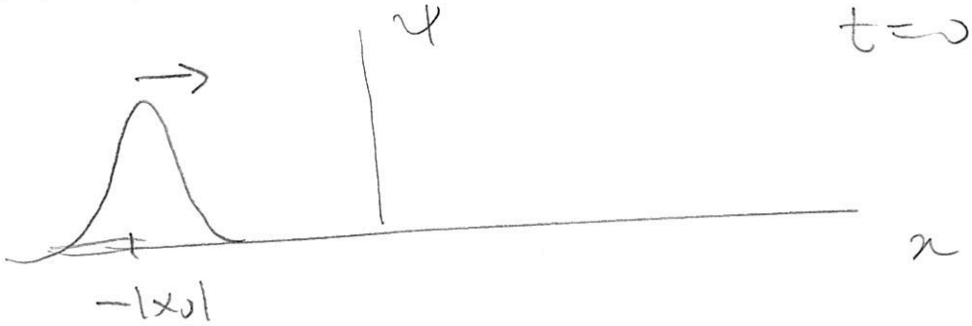
O primeiro termo contribui bastante quando

$x = x_0 = -|x_0|$

O segundo termo contribuiria p/

$$x = -x_0 - 2\theta' = |x_0| + 2|\theta'| \quad (\theta' < 0)$$

Mas como $x < 0$ nessa expressão, esse segundo termo não contribui em $t = 0$.



EVOLUÇÃO TEMPORAL

Temos que acrescentar a fase $-i\omega(k)t$ nas duas partes de $\psi(x,0)$. Expandindo em torno de $\tilde{k}$:

$$\omega(k) \simeq \omega(\tilde{k}) + \tilde{v}t \quad ; \quad \tilde{v} = \frac{\hbar \tilde{k}}{m}$$

A função de onda $\psi(x,t)$ terá agora as seguintes contribuições (Lembre que $x<0$):

$$\psi(x,t) \approx$$

$$\frac{1}{\sqrt{2\pi}} e^{i(\tilde{k}x+\tilde{\alpha}-\tilde{w}t)} \int |g(k)|\, e^{i(x-x_0-\tilde{v}t)(k-\tilde{k})}\, dk \;+$$

$$\frac{1}{\sqrt{2\pi}} e^{-i(\tilde{k}x-\tilde{\alpha}+2\tilde{\theta}+\tilde{w}t)} \int |g(k)|\, e^{-i(x+2\theta'+x_0+\tilde{v}t)(k-\tilde{k})}\, dk$$

O primeiro termo tem seu máximo p/

$$x = x_0 + \tilde{v}t = -|x_0| + \tilde{v}t$$

O segundo termo tem máximo p/

$$x = -x_0 - 2\theta' - \tilde{v}t = |x_0| + \frac{2}{\sqrt{k_0^2-\tilde{k}^2}} - \tilde{v}t$$

ANÁLISE

1) Para $t = 0$ o 1º termo contribui p/ $x \approx x_0$. O 2º termo NÃO contribui, pois o pto de máximo tem $x > 0$

2) Para $t \lesssim \frac{|x_0|}{\tilde{v}}$ só o 1º termo contribui e mostra o pacote andando p/ a direita com máximo em

$$x = -|x_0| + \tilde{v}t < 0$$

3) No intervalo

$$\frac{|x_0|}{\tilde{v}} < t < \frac{|x_0|}{\tilde{v}} + \tau_W \quad ; \quad \boxed{\tau_W = \frac{2/\tilde{v}}{\sqrt{k_0^2 - \tilde{k}^2}}}$$

Nenhum dos termos tem contribuição importante, pois os pontos onde ocorrem o máximo são positivos. Essa é a região de interferência.

4) Para $t > \frac{|x_0|}{\tilde{v}} + \tau_W$ só o segundo termo contribui e representa um pacote se movendo p/ a esquerda

O tempo $\tau_W = \frac{2m\hbar}{\tilde{p}\sqrt{p_0^2 - \tilde{p}^2}}$ = tempo de retardo

$\downarrow$ Wigner

é o tempo que a partícula fica "presa" perto de $x = 0$.

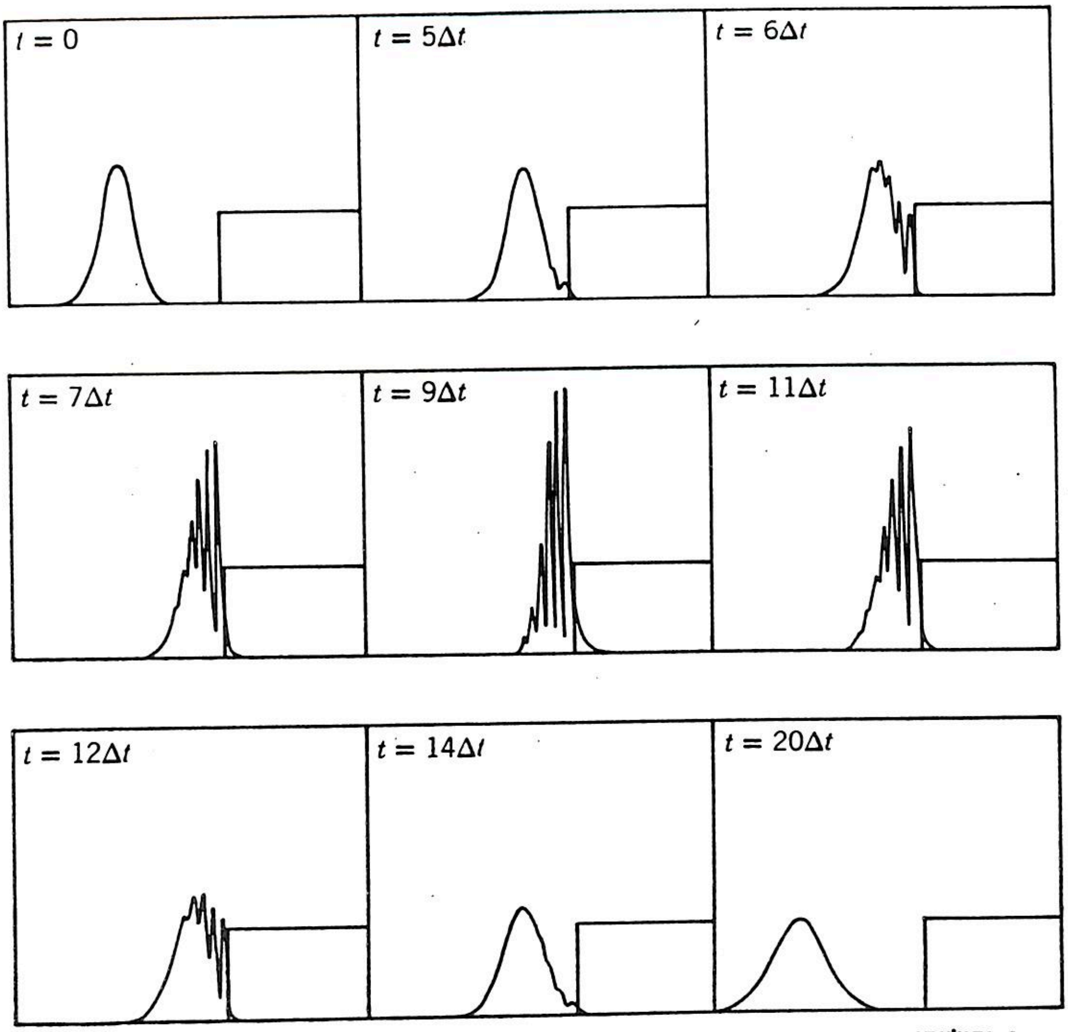


Figure 6-8 A potential step, and the probability density $\Psi^*\Psi$ for a group wave function describing a particle incident on the step with total energy less than the step height. As time evolves, the group moves up to the step, penetrates slightly into the classically excluded region, and then is completely reflected from the step. The complications of the mathematical treatment using a group are indicated by the complications of its structure during reflection.