enberg picture to be nonlinear. Since it involves an integral over all space, it can e interpreted somewhat casually as describing the effect of all the particles on a ingle one. The effective potential

$$\sum_{\sigma'} \int V(\mathbf{r}, \mathbf{r}')\psi^\dagger_{\sigma'}(\mathbf{r}', t)\psi_{\sigma'}(\mathbf{r}', t)\, d^3r' = \sum_{\sigma'} \int V(\mathbf{r}, \mathbf{r}')\rho_{\sigma'}(\mathbf{r}', t)\, d^3r'$$

an be calculated only if the solution of the equation of motion is already known, hus suggesting an iteration procedure to generate a self-consistent solution. Such echniques for solving the many-body problem are, indeed, frequently applied (see Chapter 22).

From the equation of motion (21.73) for the field operators, we can now derive he wave equation in configuration space for an n-particle system. The time-depenent wave function is defined by introducing the time into (21.56) and writing

$$\psi(\mathbf{r}_1\sigma_1, \mathbf{r}_2\sigma_2, \ldots, \mathbf{r}_n\sigma_n; t) = \frac{1}{\sqrt{n!}} \langle \mathbf{0} | \psi_{\sigma_1}(\mathbf{r}_1, t)\psi_{\sigma_2}(\mathbf{r}_2, t) \ldots \psi_{\sigma_n}(\mathbf{r}_n, t) | \Psi^{(n)} \rangle \tag{21.75}$$

We apply $i\hbar \frac{\partial}{\partial t}$ to both sides of this equation and replace the time derivatives of the eld operators by the expression on the right-hand side of (21.73). The noninteracting part of the Hamiltonian is easily seen to lead to a sum of n separate terms in he wave equation, one for each particle. The interaction term is reduced by use of he commutation or anticommutation relations for the field operators, moving $\psi^\dagger_\sigma(\mathbf{r})$ to the left in successive steps and recognizing that $\langle \mathbf{0} | \psi^\dagger_\sigma(\mathbf{r}) = 0$. When the ermutation properties of the wave function are taken into account, the wave equaon in configuration space is obtained as

$$\boxed{\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}_1\sigma_1, \mathbf{r}_2\sigma_2, \ldots \mathbf{r}_n\sigma_n; t) = \Bigg[&\sum_{j=1}^{n} H_0\left(\mathbf{r}_j, \frac{\hbar}{i}\nabla_j\right) \\ &+ \frac{1}{2} \sum_{\substack{j,\ell=1 \\ (j\neq\ell)}}^{n} V(\mathbf{r}_j, \mathbf{r}_\ell) \Bigg] \psi(\mathbf{r}_1\sigma_1, \mathbf{r}_2\sigma_2, \ldots \mathbf{r}_n\sigma_n; t) \end{aligned}} \tag{21.76}$$

This equation has the form expected for any n-particle configuration-space wave quation. The permutation symmetry of $\psi(\mathbf{r}_1\sigma_1, \mathbf{r}_2\sigma_2, \ldots, \mathbf{r}_n\sigma_n)$ is conserved as a unction of time.

Exercise 21.11. Complete the steps in the derivation of (21.76) from (21.75).

Problems

. (a) Show that if $V(r)$ is a two-particle interaction that depends only on the distance r between the particles, the matrix element of the interaction in the **k**-representation may be reduced to

$$\langle \mathbf{k}_3\mathbf{k}_4 | V | \mathbf{k}_1\mathbf{k}_2 \rangle = \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \frac{1}{(2\pi)^3} \int V(r)\, e^{-i\mathbf{q}\cdot\mathbf{r}}\, d^3r$$

where $\hbar\mathbf{q}$ is the momentum transfer $\hbar(\mathbf{k}_3 - \mathbf{k}_1)$.

(b) For this interaction, show that the mutual potential energy operator is

$$\mathcal{V} = \frac{1}{2}\iiint d^3k_1\, d^3k_2\, d^3q\; \boldsymbol{\phi}^\dagger(\mathbf{k}_1 + \mathbf{q})\boldsymbol{\phi}^\dagger(\mathbf{k}_2 - \mathbf{q})\boldsymbol{\phi}(\mathbf{k}_2)\boldsymbol{\phi}(\mathbf{k}_1)F(\mathbf{q})$$

where $F(\mathbf{q})$ is the Fourier transform of the displacement-invariant interaction.

2. Show that the diagonal part of the interaction operator $\mathcal{V}$, found in Problem 1 in the **k**-representation, arises from momentum transfers $\mathbf{q} = 0$ and $\mathbf{q} = \mathbf{k}_2 - \mathbf{k}_1$, respectively. Write down the two interaction terms and identify them as *direct* ($\mathbf{q} = 0$) and *exchange* ($\mathbf{q} = \mathbf{k}_2 - \mathbf{k}_1$) interactions. Draw the corresponding diagrams (Figure 21.1).

3. In the **k**-representation, calculate the matrix element of the interaction in Problem 1 for the screened Coulomb potential $V_0 e^{-\alpha r}/\alpha r$ and plot it as a function of q. For bosons and fermions, construct the corresponding two-particle interaction operator $\mathcal{V}$ for identical particles in terms of the creation and annihilation operators in **k**-space.

4. Defining the momentum space annihilation operator[3]

$$\boldsymbol{\phi}(\mathbf{p}) = \int \langle \mathbf{p}|\mathbf{r}\rangle \boldsymbol{\psi}(\mathbf{r})\, d^3r$$

derive the commutation (or anticommutation) relations for $\boldsymbol{\phi}(\mathbf{p})$ and $\boldsymbol{\phi}^\dagger(\mathbf{p})$. For the Bose-Einstein case, show that the mixed commutator of field operators in coordinate and momentum space is

$$[\boldsymbol{\phi}(\mathbf{p}), \boldsymbol{\psi}^\dagger(\mathbf{r})] = \langle \mathbf{p}|\mathbf{r}\rangle$$

5. In the second-quantization formalism, define the additive position and total momentum operators

$$\mathbf{r} = \int \boldsymbol{\psi}^\dagger(\mathbf{r})\mathbf{r}\boldsymbol{\psi}(\mathbf{r})\, d^3r \quad \text{and} \quad \mathbf{p} = \int \boldsymbol{\phi}^\dagger(\mathbf{p})\mathbf{p}\boldsymbol{\phi}(\mathbf{p})\, d^3p$$

and prove that for bosons their commutator is

$$[\mathbf{r}, \mathbf{p}] = i\hbar N\, \mathbf{1}$$

where N is the operator representing the total number of particles. Derive the Heisenberg uncertainty relation for position and momentum of a system of bosons, and interpret the result.

6. Local particle and current density operators at position **r** are defined in the second-quantization formalism as

$$\rho(\mathbf{r}) = \int \boldsymbol{\psi}^\dagger(\mathbf{r}')\delta(\mathbf{r}' - \mathbf{r})\boldsymbol{\psi}(\mathbf{r}')\, d^3r'$$

and

$$j(\mathbf{r}) = \frac{\hbar}{2mi}\int d^3r'\; \boldsymbol{\psi}^\dagger(\mathbf{r}')[\nabla'\delta(\mathbf{r}' - \mathbf{r}) + \delta(\mathbf{r}' - \mathbf{r})\nabla']\boldsymbol{\psi}(\mathbf{r}')$$

(a) Show that the expectation values of these operators for one-particle states are the usual expressions.

(b) Derive the formulas for the operators $\rho(\mathbf{r})$ and $j(\mathbf{r})$ in the momentum representation.

7. Two identical bosons or fermions in a state

$$\Psi^{(2)} = A\sum_{ij} c_i d_j a_j^\dagger a_i^\dagger \Psi^{(0)} = A\sum_j d_j a_j^\dagger \sum_i c_i a_i^\dagger |\mathbf{0}\rangle$$

[3]For simplicity in Problems 4–6 we suppress any spin reference to spin variables.

are said to be *uncorrelated* (except for the effect of statistics). If $\sum |c_i|^2 = \sum |d_i|^2 = 1$, determine the normalization constant A in terms of the sum $S = \sum c_i^* d_i$.

(a) In this state, work out the expectation value of an additive one-particle operator in terms of the one-particle amplitudes c_i and d_i and the matrix elements $\langle i|K|j\rangle$.

(b) Show that if $S = 0$, the expectation value is the same as if the two particles with amplitudes c_i and d_i were distinguishable.

(c) Work out the expectation value of a diagonal interaction operator in terms of c_i, d_i, and the matrix elements $\langle ij|K|k\ell\rangle = V_{ij}\delta_{ik}\delta_{j\ell}$. Show that the result is the same as for distinguishable particles if the states of the two particles do not overlap, i.e., if $c_i d_i = 0$ for all i.

A state of n identical particles (bosons or fermions) is denoted by $|\Psi^{(n)}\rangle$.

For $n = 1$, the probability of finding the particle in the one-particle basis state i is the expectation value $\langle\Psi^{(1)}|N_i|\Psi^{(1)}\rangle$. (See Exercise 21.1.)

(a) For $n = 2$, prove that the probability of finding both particles in the one-particle basis state i is the expectation value of $N_i(N_i - 1)/2$.

(b) For $n = 3$, obtain the function of N_i whose expectation value is the probability of finding all three particles in the same basis state i.

(c) For $n = 2$, show that the expectation value of $N_i N_j$ is the probability of finding the two particles in two different basis states, $i \neq j$. Prove that the probability of finding one particle in basis state i and the other particle *not* in basis state i is the expectation value of $N_i(2 - N_i)$.