where the + sign pertains to Bose-Einstein statistics and the − sign to Fermi-Dirac statistics. Also consider the Maxwell-Boltzmann limit.

In the next chapter, the formalism developed here will be applied to the derivation of the Planck distribution for photons in thermal equilibrium.

Problems

1. Consider a system of identical bosons with only two one-particle basis states, $a_{1/2}^\dagger \Psi^{(0)}$ and $a_{-1/2}^\dagger \Psi^{(0)}$. Define the Hermitian operators x, p_x, y, p_y by the relations

$$a_{1/2} = \frac{1}{\sqrt{2\hbar}}\left(cx + i\frac{p_x}{c}\right), \qquad a_{-1/2} = \frac{1}{\sqrt{2\hbar}}\left(cy + i\frac{p_y}{c}\right)$$

where c is an arbitrary real constant, and derive the commutation relations for these Hermitian operators. Express the angular momentum operator (22.6) in terms of these "coordinates" and "momenta," and also evaluate $\mathcal{J}^2$. Relate $\mathcal{J}^2$ to the square of the Hamiltonian of an isotropic two-dimensional harmonic oscillator by making the identification $c = \sqrt{m\omega}$, and show the connection between the eigenvalues of these operators.

2. (a) Using the fermion creation operators $a_{jm}^\dagger$, appropriate to particles with angular momentum j, form the closed-shell state in which all one-particle states $m = -j$ to $+j$ are occupied.

(b) Prove that the closed shell has zero total angular momentum.

(c) If a fermion with magnetic quantum number m is missing from a closed shell of particles with angular momentum j, show that, for coupling angular momenta, the hole state may be treated like a one-particle state with magnetic quantum number $-m$ and an effective creation operator $(-1)^{j-m}a_{jm}$.

3. Consider the unperturbed states $a_{nm_n}^\dagger \cdots a_{km_k}^\dagger \cdots a_{1m_1}^\dagger |\mathbf{0}\rangle$ of n spin one-half particles, each occupying one of n equivalent, degenerate orthogonal orbitals labeled by the quantum number k, and with $m_k = \pm 1/2$ denoting the spin quantum number associated with the orbital k. Show that in the space of the 2^n unperturbed states a spin-independent two-body interaction may, in first-order perturbation theory, be replaced by the effective exchange (or *Heisenberg*) Hamiltonian

$$\mathcal{H}_{\text{eff}} = -\frac{1}{\hbar^2}\sum_{k\ell} \langle k\ell|V|\ell k\rangle \mathbf{S}_k \cdot \mathbf{S}_\ell + \text{const.}$$

where $\mathbf{S}_k$ is the *localized* spin operator

$$\mathbf{S}_k = \frac{\hbar}{2}\sum_{m_k m_k'} a_{km_k}^\dagger a_{km_k'} \langle m_k|\boldsymbol{\sigma}|m_k'\rangle$$

4. For a Fermi gas of free particles with Fermi momentum p_F, calculate the ground state expectation value of the pair density operator

$$\sum_{\sigma',\sigma''} \psi_{\sigma'}^\dagger(\mathbf{r}')\psi_{\sigma''}^\dagger(\mathbf{r}'')\psi_{\sigma''}(\mathbf{r}'')\psi_{\sigma'}(\mathbf{r}')$$

in coordinate space and show that there is a repulsive interaction that would be absent if the particles were not identical. Show that there is no spatial correlation between particles of opposite spin.

5. Calculate in first order the energies of the ^1S, ^3P, and ^1D states arising from the atomic configuration p^2 (two electrons with $\ell = 1$ in the same shell). Use the multipole expansion

$$\frac{e^2}{|\mathbf{r}' - \mathbf{r}''|} = e^2 \sum_{k=0}^{\infty} \frac{4\pi}{2k+1} \gamma_k(r', r'') \sum_{q=-k}^{k} (-1)^q Y_k^q(\hat{\mathbf{r}}') Y_k^{-q}(\hat{\mathbf{r}}'')$$

for the interaction energy between the electrons, and show that the term energies may be expressed as

$$E(^1S) = E_0 + \langle \gamma_0 \rangle + \frac{10}{25} \langle \gamma_2 \rangle$$

$$E(^3P) = E_0 + \langle \gamma_0 \rangle - \frac{5}{25} \langle \gamma_2 \rangle$$

$$E(^1D) = E_0 + \langle \gamma_0 \rangle + \frac{1}{25} \langle \gamma_2 \rangle$$

where $\langle \gamma_k \rangle$ is the radial integral

$$\langle \gamma_k \rangle = e^2 \iint \gamma_k(r', r'')[R(r')]^2[R(r'')]^2 r'^2 r''^2 \, dr' \, dr''$$

6. Apply the Hartree-Fock method to a system of two "electrons" which are attracted to the coordinate origin by an isotropic harmonic oscillator potential $m\omega^2 r^2/2$ and which interact with each other through a potential $V = C(\mathbf{r}' - \mathbf{r}'')^2$. Solve the Hartree-Fock equations for the ground state and compare with the exact result and with first-order perturbation theory.