Problems

1. Consider a system of photons with fixed propagation vector in the positive z direction. The one-photon basis states are linear polarization states in x and y directions, $a_x^\dagger|\mathbf{0}\rangle$ and $a_y^\dagger|\mathbf{0}\rangle$.

(a) If $a_L^\dagger$ and $a_R^\dagger$ are the creation operators for positive and negative helicity photons, respectively, express the photon spin operator

$$S_z = \hbar(a_L^\dagger a_L - a_R^\dagger a_R)$$

in terms of the creation and annihilation operators for linear polarization.

(b) Work out the commutation relations of S_z with $a_x^\dagger$ and $a_y^\dagger$, and compare them with the commutation relations for the components of a vector operator. Show that S_z commutes with the number-of-photons operator.

(c) Construct the general two-photon state for this system. Derive the two-photon eigenstates of S_z, and interpret them in terms of the helicity of the photons.

(d) For an ensemble of unpolarized two-photon states, obtain the density matrix. How would the density matrix look if the two photons were distinguishable?

2. Prove that in a hydrogen atom the radiative transition from the $2S$ excited to the $1S$ ground state cannot occur by emission of one photon. Outline (but do not attempt to carry through in detail) the calculation of the transition rate for two-photon emission. In the dipole approximation, show that the two photons are preferentially emitted in the same direction or in opposite directions and that the angular correlation function is proportional to $1 + \cos^2\theta$, if θ is the angle between the photon momenta and if the polarization of the emitted light is not observed. Estimate the order of magnitude of the lifetime of the metastable $2S$ state.

3. Evaluate the peak value of the cross section for electric dipole absorption by a linear harmonic oscillator in its ground state, assuming that the excited state is depleted only by spontaneous emission. Use Eq. (19.134).

4. Compute the total rate of photon emission from the initial $n = 3$ level of a hydrogen atom to the $n = 2$ level (Balmer alpha line) and the mean lifetime of the atom, assuming that initially the substates of the $n = 3$ level are equally populated.