Problems

1. If **A** and **B** are proportional to the unit 4×4 matrix, derive expansion formulas for the matrix products $(\boldsymbol{\alpha} \cdot \mathbf{A})(\boldsymbol{\alpha} \cdot \mathbf{B})$ and $(\boldsymbol{\alpha} \cdot \mathbf{A})(\boldsymbol{\Sigma} \cdot \mathbf{B})$ in terms of $\boldsymbol{\alpha}$ and $\boldsymbol{\Sigma}$ matrices in analogy with formula (16.59).
2. If a field theory of massless spin one-half particles (neutrinos) is developed, so that the β matrix is absent, show that the conditions (24.30) and (24.31) are solved by 2×2 Pauli matrices, $\boldsymbol{\alpha} = \pm\boldsymbol{\sigma}$. Work out the details of the resulting *two-component* theory with particular attention to the helicity properties. Is this theory invariant under spatial reflection?
3. Develop the outlines of relativistic quantum field theory for neutral spinless bosons with mass. What modifications are indicated when the particles are charged?
4. Show that the vector operator
$$\mathbf{Q} = \beta\boldsymbol{\Sigma} + (1 - \beta)\boldsymbol{\Sigma} \cdot \hat{\mathbf{p}}\hat{\mathbf{p}}$$
satisfies the same commutation relations as $\boldsymbol{\Sigma}$ and that it commutes with the free Dirac particle Hamiltonian. Show that the eigenvalues of any component of **Q** are ± 1.
Apply the unitary transformation
$$\exp\,[i(\theta/2)\,(-p_y\,Q_x + p_x\,Q_y)/\sqrt{p_x^2 + p_y^2}]$$
to the spinors (24.92) and (24.93), and prove that the resulting spinors are eigenstates of H with sharp momentum and definite value of Q_z. Show that these states are the relativistic analogues of the nonrelativistic momentum eigenstates with "spin up" and "spin down."
5. Assume that the potential energy $-e\phi(\mathbf{r})$ in the Dirac Hamiltonian (24.175) is a square well of depth V_0 and radius a. Determine the continuity condition for the Dirac wave function ψ at $r = a$, and derive a transcendental equation for the minimum value of V_0 which just binds a particle of mass m for a given value of a.
6. Solve the relativistic Schrödinger equation for a spinless particle of mass m and charge $-e$ in the presence of the Coulomb field of a point nucleus with charge Ze. Compare the fine structure of the energy levels with the corresponding results for the Dirac electron.
7. Consider a neutral spin one-half Dirac particle with mass and with an intrinsic magnetic moment, and assume the Hamiltonian
$$H = c\boldsymbol{\alpha} \cdot \frac{\hbar}{i}\nabla + \beta mc^2 + \lambda B\beta\Sigma_z$$
in the presence of a uniform constant magnetic field along the z axis. Determine the important constants of the motion, and derive the energy eigenvalues. Show that orbital and spin motions are coupled in the relativistic theory but decoupled in a nonrelativistic limit. The coefficient λ is a constant, proportional to the gyromagnetic ratio.
8. If a Dirac electron is moving in a uniform constant magnetic field pointing along the z axis, determine the energy eigenvalues and eigenspinors.