This should not be confused with our earlier remarks concerning the invariance of the Maxwell equations (4.4.2) and the Lorentz force equation under $t \to -t$ and (4.4.3). There we were to apply time reversal to the *whole world*, for example, even to the currents in the wire that produces the **B** field!

PROBLEMS

1. Calculate the *three lowest* energy levels, together with their degeneracies, for the following systems (assume equal mass *distinguishable* particles):
   a. Three noninteracting spin $\frac{1}{2}$ particles in a box of length L.
   b. Four noninteracting spin $\frac{1}{2}$ particles in a box of length L.
2. Let $\mathscr{T}_{\mathbf{d}}$ denote the translation operator (displacement vector **d**); $\mathscr{D}(\hat{\mathbf{n}}, \phi)$, the rotation operator ($\hat{\mathbf{n}}$ and ϕ are the axis and angle of rotation, respectively); and π the parity operator. Which, if any, of the following pairs commute? Why?
   a. $\mathscr{T}_{\mathbf{d}}$ and $\mathscr{T}_{\mathbf{d}'}$ (**d** and **d**$'$ in different directions).
   b. $\mathscr{D}(\hat{\mathbf{n}}, \phi)$ and $\mathscr{D}(\hat{\mathbf{n}}', \phi')$ ($\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$ in different directions).
   c. $\mathscr{T}_{\mathbf{d}}$ and π.
   d. $\mathscr{D}(\hat{\mathbf{n}}, \phi)$ and π.
3. A quantum-mechanical state Ψ is known to be a simultaneous eigenstate of two Hermitian operators A and B which *anticommute*,

$$AB + BA = 0.$$

   What can you say about the eigenvalues of A and B for state Ψ? Illustrate your point using the parity operator (which can be chosen to satisfy $\pi = \pi^{-1} = \pi^{\dagger}$) and the momentum operator.
4. A spin $\frac{1}{2}$ particle is bound to a fixed center by a spherically symmetrical potential.
   a. Write down the spin angular function $\mathscr{Y}_{l=0}^{j=1/2, m=1/2}$.
   b. Express $(\boldsymbol{\sigma}\cdot\mathbf{x})\, \mathscr{Y}_{l=0}^{j=1/2, m=1/2}$ in terms of some other $\mathscr{Y}_l^{j, m}$.
   c. Show that your result in (b) is understandable in view of the transformation properties of the operator $\mathbf{S}\cdot\mathbf{x}$ under rotations and under space inversion (parity).
5. Because of weak (neutral-current) interactions there is a parity-violating potential between the atomic electron and the nucleus as follows:

$$V = \lambda\left[\delta^{(3)}(\mathbf{x})\mathbf{S}\cdot\mathbf{p} + \mathbf{S}\cdot\mathbf{p}\delta^{(3)}(\mathbf{x})\right],$$

   where **S** and **p** are the spin and momentum operators of the electron, and the nucleus is assumed to be situated at the origin. As a result, the ground state of an alkali atom, usually characterized by $|n, l, j, m\rangle$ actually contains very tiny contributions from other eigenstates as follows:

$$|n, l, j, m\rangle \rightarrow |n, l, j, m\rangle + \sum_{n'l'j'm'} C_{n'l'j'm'}|n', l', j', m'\rangle.$$

On the basis of symmetry considerations *alone*, what can you say about (n', l', j', m'), which give rise to nonvanishing contributions? Suppose the radial wave functions and the energy levels are all known. Indicate how you may calculate $C_{n'l'j'm'}$. Do we get further restrictions on (n', l', j', m')?

6. Consider a symmetric rectangular double-well potential:

$$V = \begin{cases} \infty & \text{for } |x| > a + b; \\ 0 & \text{for } a < |x| < a + b; \\ V_0 > 0 & \text{for } |x| < a. \end{cases}$$

Assuming that V_0 is very high compared to the quantized energies of low-lying states, obtain an approximate expression for the energy splitting between the two lowest-lying states.

7. a. Let $\psi(\mathbf{x}, t)$ be the wave function of a spinless particle corresponding to a plane wave in three dimensions. Show that $\psi^*(\mathbf{x}, -t)$ is the wave function for the plane wave with the momentum direction reversed.
   b. Let $\chi(\hat{\mathbf{n}})$ be the two-component eigenspinor of $\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}$ with eigenvalue $+1$. *Using the explicit form of* $\chi(\hat{\mathbf{n}})$ (in terms of the polar and azimuthal angles β and γ that characterize $\hat{\mathbf{n}}$) verify that $-i\sigma_2\chi^*(\hat{\mathbf{n}})$ is the two-component eigenspinor with the spin direction reversed.
8. a. Assuming that the Hamiltonian is invariant under time reversal, prove that the wave function for a spinless nondegenerate system at any given instant of time can always be chosen to be real.
   b. The wave function for a plane-wave state at $t = 0$ is given by a complex function $e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}$. Why does this not violate time-reversal invariance?
9. Let $\phi(\mathbf{p}')$ be the momentum-space wave function for state $|\alpha\rangle$, that is, $\phi(\mathbf{p}') = \langle\mathbf{p}'|\alpha\rangle$. Is the momentum-space wave function for the time-reversed state $\Theta|\alpha\rangle$ given by $\phi(\mathbf{p}')$, $\phi(-\mathbf{p}')$, $\phi^*(\mathbf{p}')$, or $\phi^*(-\mathbf{p}')$? Justify your answer.
10. a. What is the time-reversed state corresponding to $\mathscr{D}(R)|j, m\rangle$?
    b. Using the properties of time reversal and rotations, prove

$$\mathscr{D}^{(j)*}_{m'm}(R) = (-1)^{m-m'}\mathscr{D}^{(j)}_{-m',-m}(R).$$

    c. Prove $\Theta|j, m\rangle = i^{2m}|j, -m\rangle$.
11. Suppose a spinless particle is bound to a fixed center by a potential $V(\mathbf{x})$ so asymmetrical that no energy level is degenerate. Using time-

reversal invariance prove

$$\langle \mathbf{L} \rangle = 0$$

for any energy eigenstate. (This is known as **quenching** of orbital angular momentum.) If the wave function of such a nondegenerate eigenstate is expanded as

$$\sum_l \sum_m F_{lm}(r) Y_l^m(\theta, \phi),$$

what kind of phase restrictions do we obtain on $F_{lm}(r)$?

12. The Hamiltonian for a spin 1 system is given by

$$H = AS_z^2 + B\left(S_x^2 - S_y^2\right).$$

Solve this problem *exactly* to find the normalized energy eigenstates and eigenvalues. (A spin-dependent Hamiltonian of this kind actually appears in crystal physics.) Is this Hamiltonian invariant under time reversal? How do the normalized eigenstates you obtained transform under time reversal?