SOLUÇÃO DO PROBLEMA 7

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \alpha\,\delta(x)$$

Estado ligado com $E<0$

$$\psi(x) = A \begin{cases} e^{-kx} & x>0 \\ e^{kx} & x<0 \end{cases} \qquad k = \sqrt{\frac{2m|E|}{\hbar^2}}$$

Descontinuidade em $x=0$: $-\frac{\hbar^2}{2m}\left[\psi'(0^+) - \psi'(0^-)\right] = \alpha\,\psi(0)$

$$-\frac{\hbar^2}{2m}\left[-kA - kA\right] = \alpha A \longrightarrow \frac{\hbar^2 k}{m} = \alpha \qquad \boxed{k = \frac{m\alpha}{\hbar^2}}$$

$$|E| = \frac{\hbar^2 k^2}{2m} \Longrightarrow \boxed{E = -\frac{m\alpha^2}{2\hbar^2}}$$

Normalização

$$1 = A^2\int_{-\infty}^{0} e^{2kx}\,dx + A^2\int_{0}^{\infty} e^{-2kx}\,dx = A^2\left[\frac{1}{2k} + \frac{1}{2k}\right] = \frac{A^2}{k}$$

$$A = \sqrt{k} = \sqrt{\frac{m\alpha}{\hbar^2}}$$

$$\psi_e(x) = \sqrt{\frac{m\alpha}{\hbar^2}} \begin{cases} e^{-\frac{m\alpha x}{\hbar^2}} & x>0 \\ e^{\frac{m\alpha x}{\hbar^2}} & x<0 \end{cases}$$

Estados de Espalhamento, $E>0$

$$\varphi_k(x) = A \begin{cases} e^{ikx} + r e^{-ikx} & x<0 \\ t\, e^{ikx} & x>0 \end{cases}$$

continuidade: $1+r=t$

descontinuidade: $-\frac{\hbar^2}{2m}\left[(ikt A)-(ik-ikr)A\right]=\alpha t$

$$ikt+ik(r-1) = -\frac{2m\alpha}{\hbar^2}t \equiv -k_0^2 t \qquad \boxed{k_0^2=\frac{2m\alpha}{\hbar^2}}$$

$$t(k_0^2+ik) = ik(1-r) = ik(2-t)$$

$$t(k_0^2+2ik) = 2ik \rightarrow t=\frac{2ik}{2ik+k_0^2} \quad \text{ou}$$

$$\boxed{t=\frac{k}{k-iR}} \quad ; \quad R=k_0^2/2 = m\alpha/\hbar^2$$

$$r=t-1 \rightarrow \boxed{r=\frac{iR}{k-iR}}$$

Normalização

- $J_+ \equiv \lim_{\varepsilon\to 0}\int_0^\infty e^{ix(p+i\varepsilon)}\,dx = \frac{-1}{i(p+i\varepsilon)} = \frac{\varepsilon}{p^2+\varepsilon^2}+\frac{ip}{p^2+\varepsilon^2}$

$$= \pi\,\delta(p) + i\,\mathcal{P}\left(\frac{1}{p}\right)$$

delta de Dirac, resultado p/ $p=0$

parte principal, resultado quando $p\neq 0$.

$$\cdot\ J_- \equiv \lim_{\varepsilon \to 0} \int_{-\infty}^{0} e^{ip(x-i\varepsilon)}\, dx = \pi\delta(p) - i\mathcal{P}\left(\frac{1}{p}\right)$$

$$I = A^2 \int_{-\infty}^{0} \left[e^{-ik'x} + R^*(k')e^{ik'x}\right]\left[e^{ikx} + R(k)e^{-ikx}\right] dx$$
$$+ A^2 \int_{0}^{\infty} t^*(k')\,t(k)\, e^{i(k-k')x}\, dx$$

com k e $k' > 0$ (duas ondas incidentes da esquerda).

$$\frac{I}{A^2} = \int_{-\infty}^{0} \left[e^{i(k-k')x} + R^*(k')e^{i(k+k')x} + R(k)e^{-i(k+k')x} + R(k)R^*(k')e^{-i(k-k')x}\right]$$
$$+ \int_{0}^{\infty} t^*(k')\,t(k)\, e^{i(k-k')x}\, dx$$

$$= \left[\pi\delta(k-k') - i\mathcal{P}\left(\frac{1}{k-k'}\right)\right] + R^*(k')\left[\pi\delta(k+k') - i\mathcal{P}\left(\frac{1}{k+k'}\right)\right]$$
$$+ R(k)\left[\pi\delta(k+k') + i\mathcal{P}\left(\frac{1}{k+k'}\right)\right] + R^*(k')R(k)\left[\pi\delta(k-k') + i\mathcal{P}\left(\frac{1}{k-k'}\right)\right]$$
$$+ t^*(k')\,t(k)\left[\pi\delta(k-k') + i\mathcal{P}\left(\frac{1}{k-k'}\right)\right]$$

Como $k, k' > 0$, $\delta(k+k') = 0$. Usando $t = 1 + R$

$$t(k)\,t^*(k') = 1 + R(k) + R^*(k') + R(k)R^*(k') \ ;$$

$$\frac{I}{A^2} = 2\pi\,\delta(k-k') - i\,\mathcal{P}\left(\frac{1}{k-k'}\right)\left[1 - 2R^*(k')R(k) - R^*(k') - R(k) - 1\right]$$

$$-i\,\mathcal{P}\left(\frac{1}{k+k'}\right)\left[R^*(k') - R(k)\right]$$

$$R(k) + R^*(k') + 2R(k)R^*(k') = \frac{iR}{k-iR} - \frac{iR}{k'+iR} + \frac{2R^2}{(k-iR)(k'+iR)}$$

$$= \frac{iR(k'-k+2iR) + 2R^2}{(k-iR)(k'+iR)}$$

$$= \frac{-iR(k-k')}{(k-iR)(k'+iR)}$$

$$R(k) - R^*(k') = \frac{iR}{k-iR} + \frac{iR}{k'+iR} = \frac{(k+k')\,iR}{(k-iR)(k'+iR)}$$

$$\Rightarrow \frac{R(k) + R^*(k') + 2R(k)R^*(k')}{k-k'} + \frac{R(k) - R^*(k')}{k+k'} = 0$$

e $\quad I = 2\pi A^2\,\delta(k-k') \equiv \delta(k-k')$

$$\Rightarrow A = \frac{1}{\sqrt{2\pi}} \quad e$$

$$\varphi_k(x) = \frac{1}{\sqrt{2\pi}}\begin{cases} e^{ikx} + R(k)\,e^{-ikx} \\ t(k)\,e^{ikx} \end{cases}$$

Probabilidade de Transição $\varphi_e(x) \to \varphi_k(x)$

$$\mathcal{P} = \frac{\pi}{2\hbar}\rho(E_f)\,|\langle \varphi_{k_f}|W^0|\varphi_e\rangle|^2$$

onde $E_f = \dfrac{\hbar^2 k_f^2}{2m} = E_i + \hbar\omega$

$$W = -q\mathcal{E}X\sin\omega t = W^0\sin\omega t$$

$$E = \frac{\hbar^2 k^2}{2m} \qquad dE = \frac{\hbar^2 k}{m}dk = \frac{\hbar^2}{m}\frac{\sqrt{2mE}}{\hbar}dk$$

$$= \hbar\sqrt{\frac{2E}{m}}\,dk \qquad dk = \underbrace{\sqrt{\frac{m}{2E}}\frac{1}{\hbar}}_{\rho(E)}dE$$

$$\mathcal{P} = \frac{\pi}{2\hbar}\sqrt{\frac{m}{2E_f}}\frac{1}{\hbar}q^2\mathcal{E}^2\,|\underbrace{\langle \varphi_{k_f}|X|\varphi_e\rangle}_{U}|^2$$

$$U = \int_{-\infty}^{0} x\left(e^{-ikx} + R^*(k)e^{ikx}\right)e^{Rx}dx + \int_0^{\infty} x\,t^*(k)\,e^{-ikx}e^{-Rx}dx$$

$$\cdot \int_{-\infty}^{0} x e^{ipx+Rx} = \int_{-\infty}^{0} x e^{x(R+ip)}dx \quad ; \quad x = u \quad dx = du \quad e^{x(R+ip)}dx = dv \quad v = \frac{e^{x(R+ip)}}{R+ip}$$

$$= \underbrace{\left.\frac{xe^{x(R+ip)}}{R+ip}\right|_{-\infty}^{0}}_{=0} - \frac{1}{R+ip}\int_{-\infty}^{0} e^{x(R+ip)}dx$$

$$= -\frac{1}{(R+ip)^2}$$

$$\int_0^\infty x e^{-ikx-Rx}\,dx = +\int_0^{-\infty} x e^{ikx+Rx}\,dx = -\int_{-\infty}^0 x e^{ikx+Rx}$$

$$= \frac{1}{(R+ik)^2}$$

$$U = \frac{-1}{(R-ik)^2} - \frac{R^*(k)}{(R+ik)^2} + \frac{t^*(k)}{(R+ik)^2} \quad ; \quad t^* = R^*+1$$

$$= \frac{-1}{(R-ik)^2} + \frac{1}{(R+ik)^2} = \frac{(R-ik)^2-(R+ik)^2}{(R^2+k^2)^2} = \frac{-4iRk}{(k^2+R^2)^2}$$

$$|U|^2 = \frac{16k^2R^2}{(k^2+R^2)^2} = 16|R|^2|t|^2$$

$$\Gamma = \frac{\pi}{2\hbar^2}\sqrt{\frac{m}{2E_f}}\, q^2\mathcal{E}^2 \frac{16k_f^2R^2}{(k_f^2+R^2)^2} = \frac{8\pi q^2\mathcal{E}^2 m}{\hbar^3}\frac{k_f R^2}{(k_f^2+R^2)^2}$$

$$R^2 = \frac{m^2\alpha^2}{\hbar^4}$$

$$E_f = \frac{\hbar^2k_f^2}{2m} = E_i + \hbar\omega = \frac{-m\alpha^2}{2\hbar^2} + \hbar\omega = -\frac{\hbar^2R^2}{2m} + \hbar\omega$$

$$k_f^2 = -R^2 + \frac{2m\omega}{\hbar} \quad \rightarrow \quad k_f^2 + R^2 = \frac{2m\omega}{\hbar}$$

$$\Gamma = \frac{2\pi q^2\mathcal{E}^2}{m\hbar}\,\frac{R^2\sqrt{\frac{2m\omega}{\hbar} - R^2}}{\omega^2}$$

$$\omega_0 \equiv \frac{\hbar k^2}{2m} = \frac{\hbar}{2m}\frac{m^2\alpha^2}{\hbar^4} = \frac{m\alpha^2}{2\hbar^3}$$

$$P = \frac{2\pi q^2 \mathcal{E}^2 R^2}{m\hbar\omega^2}\sqrt{\frac{2m}{\hbar}(\omega-\omega_0)}$$

$$P = \frac{A}{\omega^2}\sqrt{B(\omega-\omega_0)}$$

$$\frac{\partial P}{\partial \omega} = \frac{-2A}{\omega^3}\sqrt{B(\omega-\omega_0)} + \frac{A}{2\omega^2}\frac{B}{\sqrt{B(\omega-\omega_0)}} \equiv 0$$

$$-4\cancel{A}(\cancel{B})(\omega-\omega_0) + \cancel{A}\cancel{B}\omega = 0$$

$$\omega = 4\omega - 4\omega_0 \qquad 3\omega = 4\omega_0$$

$$\omega = 4\omega_0/3$$