FI 105 - Test 1 - 2015

Problem 1: Vortices electrostatic representation

Here we will develop a representation for the vortices singularities in which the φ field plays the role of point charges in a 2D electrostatic analogy. (a) We define the "displacement field" $\vec{D}$ as

$$\vec{D} = \vec{\nabla}\varphi \times \hat{z}$$

Show that

$$\vec{\nabla} \cdot \vec{D} = 2\pi k \delta^2(r)$$

(b) We write the "dielectric constant" as

$$\epsilon = \frac{1}{2\pi\rho_s}$$

and we define the electric field as

$$\vec{D} = \epsilon \vec{E}$$

Show that for the xy model we have

$$U = \frac{1}{4\pi} \int d^2 r \vec{E} \cdot \vec{D}$$

(c) Show that the electric potential is

$$V(r) = -\frac{k}{\epsilon} \ln\left(\frac{r}{a}\right)$$

We have therefore the statistical mechanics of a collection of interacting vortices equivalent of a 2D plasma of charges with potential energy

$$U = -\frac{1}{\epsilon} \sum_{i<j} k_i k_j \ln \left| \frac{\vec{r}_i - \vec{r}_j}{a} \right|$$

(d) Consider now a bound pair of vortices. From general gauge invariance considerations we can show we have a supercurrent given by

$$\vec{J}_s = \frac{2e}{\hbar} \rho_s \vec{\nabla}\varphi$$

Show that we have a force equal

$$\vec{F} = k\frac{\hbar}{2e}\vec{J}_s \times \hat{z}$$

Show that this force is opposite in sign for each member of a bound pair of vortices and the pair polarize. This polarization is exactly like the electric polarization in a medium and contributes to the renormalization of the dielectric constant. This is equivalent do a decrease in the spin stifness. Estimate this effect.

Reference: S.M. Girvin, *The Kosterlitz-Thouless Phase Transition*, Boulder Series, 2000.

2) Kinks: The Bogomolnyi method for Z_2 kink

Consider the functional form for the energy of a system as

$$E = \int dx \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + V(\phi)\right]$$

Writing the energy functional into a "whole square" form, show that for boundary values with fixed values for $\phi(\pm\infty)$, the energy is minimized if

$$\frac{\partial\phi}{\partial t} = 0$$

and

$$\frac{\partial\phi}{\partial x} = \mp\sqrt{2V(\phi)} = 0$$

and that the minimum value of the energy is

$$E_{min} = \pm\int_{\phi(-\infty)}^{\phi(+\infty)} d\phi' \sqrt{2V(\phi')}$$

(b) Consider now the ϕ^4 model, with

$$V(\phi) = \frac{\lambda}{4}(\eta^2 - \phi^2)^2$$

Show that

$$\phi(x) = \eta \tanh\left(\sqrt{\frac{\lambda}{2}}\eta x\right)$$

and calculate the energy of the kink in this case.

(c) Consider now the sine-Gordon model, with a Lagrangian given by

$$L = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{\alpha}{\beta^2}(1 - \cos(\beta\phi))$$

This is a scalar field theory in a 1+1 space-time dimension. It is invariant under $\phi \to \phi + 2\pi n$, where n is any integer. We have therefore a Z symmetry. The vacua are given by $\phi = 2\pi n/\beta$, labeled by the integer n.

Use the Bogomolnyi method and find $\phi(x)$ and the energy E of the kink.

Reference: T. Vachaspati, **Kinks and Domain Walls**, Cambridge University Press (2006).