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Usando $n = \frac{N}{V}$ ; $p_F = \left[\frac{6\pi^2}{g}\right]^{1/3} \hbar\, n^{1/3}$

$$\epsilon_F = \frac{p_F^2}{2m} = \left(\frac{6\pi^2}{g}\right)^{2/3} \frac{\hbar^2}{2m} n^{2/3} \qquad T_F = \frac{\epsilon_F}{k} \; ; \; k_F = \left(\frac{6\pi^2 N}{gV}\right)^{1/3}$$

em metais típicos, $T_F \sim 8 \times 10^4 \gg 300\,K$ (temperatura ambiente) $\Rightarrow$ gás degenerado.

Como $N = \frac{gV}{h^3} \frac{4\pi}{3} p_F^3$

$$E = \frac{gV}{2\pi^2 \hbar^3} \int_0^{p_F} p^2 \left(\frac{p^2}{2m}\right) dp = \frac{gV}{2\pi^2 \hbar^3} \frac{p_F^5}{10m} = \frac{3}{5} N \epsilon_F$$

ou $E = \left(\frac{6\pi^2}{g}\right)^{2/3} \frac{3N\hbar^2}{10m} n^{2/3}$

Eq. de estado:

$$\epsilon_p = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$$

$$\sum_i \rightarrow \frac{gV}{(2\pi)^3} \int d^3k \Rightarrow \frac{gV}{(2\pi)^3} 4\pi k^2 dk = \frac{gV}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\epsilon}\, d\epsilon$$

$$-\frac{\Omega}{kT} = \frac{PV}{kT} = +\frac{gV}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty d\epsilon \sqrt{\epsilon} \ln\left[1 + e^{\beta(\mu-\epsilon)}\right]$$

Integrando por partes:

$$PV = \frac{gV}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{2}{3} \int_0^\infty d\epsilon \frac{\epsilon^{3/2}}{e^{\beta(\epsilon-\mu)} + 1}$$

mas

$$E = \sum_i n_i \epsilon_i = \frac{gV}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty d\epsilon \frac{\epsilon^{3/2}}{e^{\beta(\epsilon-\mu)} + 1}$$

$\Rightarrow PV = \frac{2}{3} E \quad \Rightarrow$ p/ $T = 0$ e $g = 2$ ; $P = (3\pi^2)^{2/3} \frac{\hbar^2}{5m} n^{5/3}$

$\frac{E}{N} \propto n^{2/3}$  $V \sim \frac{e^2}{d} \propto n^{1/3}$

$\Rightarrow$ gás denso, en. cinética >> potencial
novamente em contradição c/ o gas clássico

Gas a baixas Temperaturas.
Deduzimos que $\left(x \equiv (\epsilon - \mu)/kT\right)$

$$PV = \frac{2}{3}\frac{gV}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}(kT)^{5/2}\int_{-\mu/kT}^{\infty} dx \frac{(x + \mu/kT)^{3/2}}{e^x + 1}$$

$$\alpha = \frac{\mu}{kT}$$

$$I(\alpha) = \int_{-\alpha}^{\infty} dx \frac{(x+\alpha)^{3/2}}{e^x + 1} = \int_{-\alpha}^{0} + \int_{0}^{\infty}$$

Façamos $x \to -x$ na 1ª integral e usemos $(1+e^{-x})^{-1} \equiv 1 - (e^x + 1)^{-1}$

$$I(\alpha) = \int_0^{\alpha} dx\, (\alpha - x)^{3/2} + \int_0^{\infty} dx \frac{(\alpha+x)^{3/2} - (\alpha - x)^{3/2}}{e^x + 1}$$
$$+ \int_{\alpha}^{\infty} dx \frac{(\alpha - x)^{3/2}}{e^x + 1}$$

$\alpha = \frac{\mu}{kT} \to \infty$ se $T \to 0 \Rightarrow$ última $\int$ é
exponencialmente pequena e
$$(\alpha + x)^{3/2} - (\alpha - x)^{3/2} \simeq 3x\alpha^{1/2} + O(\alpha^{-3/2})$$

Usando que $\displaystyle\int_0^\infty dx \frac{x^{n-1}}{e^x+1} = (1-2^{1-n})\Gamma(n)\zeta(n) \quad n>0$

$$\Gamma(z) = \int_0^\infty dt\, e^{-t} t^{z-1} \; ; \; \mathrm{Re}\, z > 0$$

$$\zeta(z) = \sum_{p=1}^{\infty} p^{-z} \; ; \; \mathrm{Re}\, z > 1$$

$$I(\alpha) \approx \frac{2}{5}\alpha^{5/2} + \frac{\pi^2}{4}\alpha^{1/2} + \dots = \left(\frac{1}{kT}\right)^{5/2}\left[\frac{2}{5}\mu^{5/2} + (kT)^2\frac{\pi^2}{4}\mu^{1/2} + \dots\right]$$

$$\Rightarrow PV = \frac{gV}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{2}{3}\left[\frac{2}{5}\mu^{5/2} + (kT)^2\frac{\pi^2}{4}\mu^{1/2} + \dots\right]$$

$$N = \left[\frac{\partial(PV)}{\partial\mu}\right]_{TV} = \frac{gV}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{2}{3}\left[\mu^{3/2} + (kT)^2\frac{\pi^2}{8\mu^{1/2}} \dots\right]$$

$$N = \frac{gV}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{2}{3}\mu^{3/2}\left[1 + (kT)^2\frac{\pi^2}{8\mu^2}\right] \dots$$

mas $\epsilon_F = \mu(T=0) = \left(\frac{6\pi^2}{g}\right)^{2/3}\frac{\hbar^2}{2m}\left(\frac{N}{V}\right)^{2/3}$

$$\Rightarrow \frac{N}{V} = \epsilon_F^{3/2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{g}{4\pi^2}\frac{2}{3}$$

$$\Rightarrow \epsilon_F^{3/2} = \mu^{3/2}\left[1 + (kT)^2\frac{\pi^2}{8\mu^2}\right]$$

$$\Rightarrow \mu = \epsilon_F\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right]^{-2/3}$$

$$\Rightarrow \mu \approx \epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2 + \dots\right]$$

$$S(T,V,\mu) = \frac{\partial (PV)}{\partial T}\Big|_{V\mu} = \frac{gV}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{2}{3}\left[\frac{2\pi^2}{4}k^2T\mu^{1/2}+\dots\right]$$

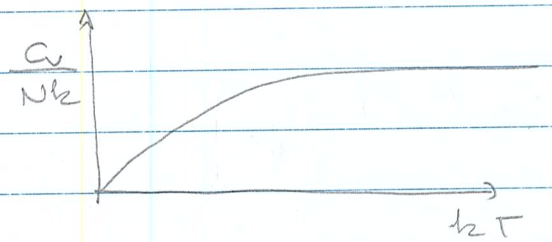
Substituindo expressão anterior p/ $\mu$

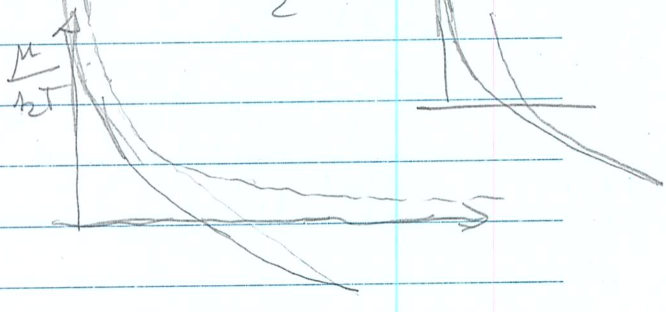
$$S(T,V,N) = \frac{Nk^2T}{\epsilon_F}\frac{\pi^2}{2}$$

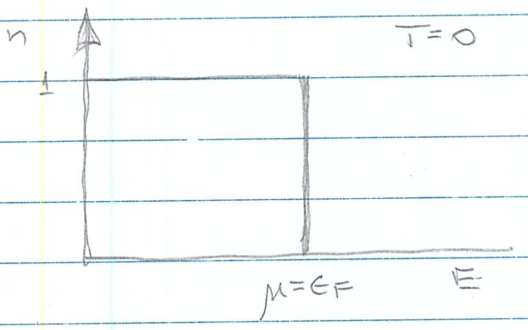
$$C_V = T\frac{\partial S}{\partial T}\Big|_{VN} = \frac{\pi^2}{2}Nk\frac{kT}{\epsilon}.$$

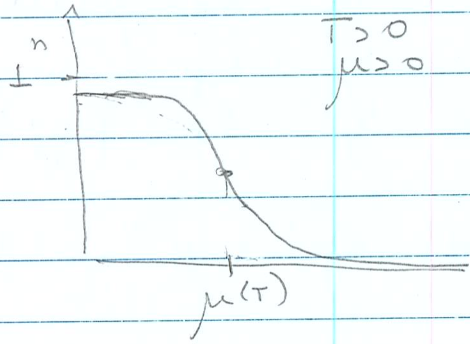
$$\Rightarrow C_V = \frac{mk^2}{\hbar^2}\left(\frac{g\pi}{6}\right)^{2/3} NT\, n^{2/3} \;;\; T \to 0$$

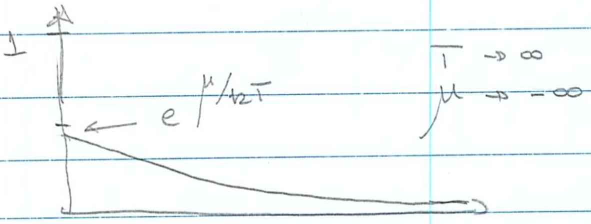
Por outro lado, classicamente: $C_V \to \frac{3}{2}Nk$.




$$\mu(T) \to kT\ln\left(\frac{N}{gV}\lambda_T^3\right)$$

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## Gás ideal de Bósons:

O nº de ocupação bosônico é

$n_i = \dfrac{1}{e^{\beta(\epsilon_i - \mu)} - 1}$ e no limite contínuo

Temos

$$PV = \frac{2}{3} E = \frac{2}{3} \frac{gV}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty d\epsilon \frac{\epsilon^{3/2}}{e^{\beta(\epsilon-\mu)} - 1}$$

$$\text{e} \quad \frac{N}{V} = \frac{g}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty d\epsilon \frac{\sqrt{\epsilon}}{e^{\beta(\epsilon-\mu)} - 1}$$

e a única diferença com respeito a férmions vem da distribuição $n_i$ ou $n(\epsilon)$.

Esta fórmula é válida para bósons massivos pois $\epsilon = \dfrac{\hbar^2 k^2}{2m}$.

No caso de bósons não massivos: fótons, fónons etc.

Temos $\epsilon = pc = c|\vec{p}|$.

Consideremos o caso de fótons (radiação de corpo negro) degenerescência: $g = 2$ (dois estados de polarização)

$$\Rightarrow E(V,T) = \sum_{\vec{p}} c|\vec{p}|\, n(\vec{p}) = \frac{2V}{h^3} \int d^3p \frac{cp}{e^{\beta c|p|} - 1}$$

(Lembre-se que $\mu = 0$!). Fazendo $v = \beta c p$

$$\epsilon(T) = \frac{E(V,T)}{V} = \frac{(kT)^4}{\pi^2 \hbar^3 c^3} \int_0^\infty dv \frac{v^3}{e^v - 1} = \frac{\pi^2 (kT)^4}{15 \hbar^3 c^3}$$

onde $B_n = \displaystyle\int dx \frac{x^n}{e^x - 1}$ é tal que $B_1 = \dfrac{\pi^2}{6}$ $B_2 = 2\zeta(3)$ e $B_3 = \dfrac{\pi^4}{15}$

$= \Gamma(n+1)\zeta(n+1)$

Para o gás ultra-relativístico: $PV = \frac{E}{3}$

$\Rightarrow P = \frac{\epsilon}{3}$

Então: $\epsilon(T) = \sigma' T^4$ $\quad \sigma' = \frac{\pi^2 k^4}{15 \hbar^3 c^3}$

e $P = \frac{1}{3} \sigma' T^4$

Em termos da frequência:

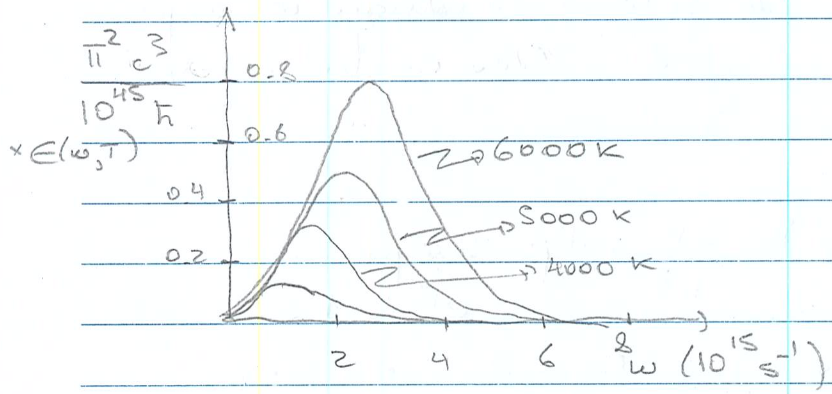
$$\epsilon(T) = \int_0^\infty \epsilon(\omega, T)\, d\omega$$

como $\nu = \beta c p = \beta c \hbar k = \beta \hbar \omega$ pois $\omega = ck$

$$\epsilon(\omega, T) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/kT} - 1}$$

escrita pela 1ª vez em 1900 por Planck.

"Lei de Planck da radiação do corpo negro"




Análise dimensional: $\epsilon(T) = A \hbar^{-3} c^{-3} (kT)^4$

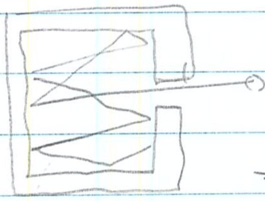
sem $\hbar$: impossível

Mas $\epsilon(\omega, T)$ sem $\hbar$

pode ser $\epsilon(\omega, T) = B c^{-3} (kT) \omega^2$

que diverge! (limite clássico da distribuição)

Radiação do corpo negro:



$\phi(T) \to$ fluxo de energia = en./área tempo

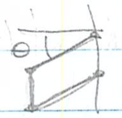
# de fótons p/ volume em $d^3\vec{p}$ $= f(\vec{p})$

$$f(\vec{p}) = \frac{2}{h^3}\frac{1}{e^{\beta c p}-1} \quad \text{pois} \quad N = 2\sum_i n_i = 2\frac{V}{(2\pi)^3}\int d^3k\, n(k)$$

$$= 2\frac{V}{(2\pi)^3\hbar^3}\int d^3p\, n(p)$$

como $f(\vec{p}) = f(p)$

$$\epsilon(T) = \int d^3p\, f(p)\, pc = 4\pi c\int_0^\infty dp\, f(p)\, p^3$$



$$\epsilon_p v_z = c^2 p\cos\theta \qquad (|\vec{J}_E|)$$

$$\phi(T) = \int_{p_z>0} d^3p\, f(\vec{p})\, p\cos\theta$$

$$= \int_0^\infty dp\, c^2 p^3 f(p)\left[2\pi\underbrace{\int_0^{\pi/2} d\theta\,\sin\theta\cos\theta}_{1/2}\right]$$

$$= \frac{c}{4}\epsilon(T) = \sigma T^4 \quad \text{onde} \quad \sigma = \frac{\pi^2 k^4}{60\hbar^3 c^2}$$

$\sigma = 5.67\times 10^{-8}$ watt $m^{-2}\,K^{-4}$

fluxo / frequência $= \dfrac{\omega^3}{e^{\hbar\omega/kT}-1} \propto \dfrac{\nu^3}{e^{\nu}-1}$ ; $\nu = \dfrac{\hbar\omega}{kT}$

máx da curva: $\nu_m = 2.82$ ou $\omega_m = 2.82\dfrac{kT}{\hbar}$

$\Rightarrow \dfrac{\hbar\omega_{1m}}{kT_1} = \dfrac{\hbar\omega_{2m}}{kT_2} \to$ Lei do deslocamento de Wien