1.5) Hidrodinâmica de fluidos simples.

Fluido simples $\Rightarrow$ monomolecular sem estrutura.
ex: argônio { líquido (fluido denso) / gás (fluido diluído)

5 grandezas conservadas: massa, energia e 3 componentes do momento linear

$\rho = mn \rightarrow$ densidade de massa, $\epsilon \rightarrow$ de energia
$\vec{g}$ de momento linear

Conservação de massa:
Elemento do fluido com velocidade $\vec{u}(\vec{r},t)$

$$\Rightarrow \frac{\partial \rho(\vec{r},t)}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0 \quad \text{ou} \quad \frac{\partial \rho}{\partial t} + \partial_\alpha(\rho u_\alpha) = 0$$

(convenção de Einstein) que pode ser reescrita como:

$$\frac{\partial \rho}{\partial t} + u_\alpha \partial_\alpha \rho + \rho\, \partial_\alpha u_\alpha = 0 \Rightarrow \frac{D\rho}{Dt} + \rho(\vec{\nabla}\cdot\vec{u}) = 0$$

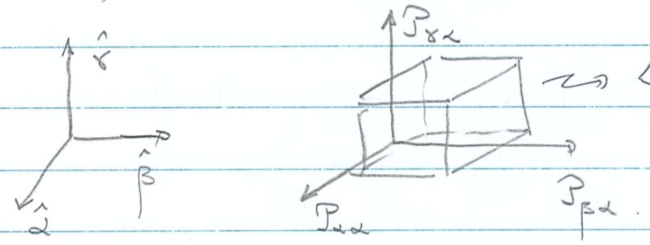
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_\alpha \partial_\alpha = \frac{\partial}{\partial t} + (\vec{u}\cdot\vec{\nabla})$$

No referencial do laboratório

$$du_\alpha = u_\alpha(\vec{r} + d\vec{r}, t + dt) - u_\alpha(\vec{r}, t)$$
$$= \frac{\partial u_\alpha}{\partial t} dt + dt \frac{d\vec{r}}{dt}\cdot\vec{\nabla} u_\alpha = \frac{Du_\alpha}{Dt} dt$$

$(\vec{u}\cdot\vec{\nabla})$ é o chamado termo de <u>convecção</u>

Conservação do momento linear.



$P_{\alpha\beta}$ ← direção da área onde é aplicada

↑ direção da força

$$\Rightarrow \rho\,\Delta V \frac{Du_\alpha}{Dt} = -\int_S P_{\alpha\beta}\, d^2 S_\beta = -\int_{\Delta V} d^3 r\, \partial_\beta P_{\alpha\beta}.$$

$$\Rightarrow \boxed{\rho \frac{Du_\alpha}{Dt} = -\partial_\beta P_{\alpha\beta}.}$$

No referencial de Galileo, local e instantâneo, no elemento de fluido ⇒ fluido em repouso
$\vec{u} = 0$

$$\Rightarrow \rho \frac{\partial u_\alpha}{\partial t} = -\partial_\beta P_{\alpha\beta}$$

mas $\rho \frac{\partial u_\alpha}{\partial t} = \frac{\partial}{\partial t}(\rho u_\alpha) - \frac{\partial \rho}{\partial t} \cancel{u_\alpha}^{\,0}$

$$\Rightarrow \frac{\partial}{\partial t}(\rho u_\alpha) = \frac{\partial g_\alpha}{\partial t} = -\partial_\beta P_{\alpha\beta} \quad ; \; g_\alpha \to \text{densidade de momento linear}$$

ou $\boxed{\frac{\partial g_\alpha}{\partial t} + \partial_\beta P_{\alpha\beta} = 0}$

$P_{\alpha\beta}$ → corrente associada a g_α

Em geral, Tensor $T_{\alpha\beta}$ tq $T'_{\alpha\beta} = P_{\alpha\beta}$ no ref. em repouso

Se o fluido é incompressível e $P_{\alpha\beta} = \delta_{\alpha\beta} P$

A derivada convectiva é

$$\rho \frac{D u_\alpha}{Dt} = \rho \frac{\partial u_\alpha}{\partial t} + \rho (\vec{u} \cdot \vec{\nabla}) u_\alpha$$

Incompressibilidade:

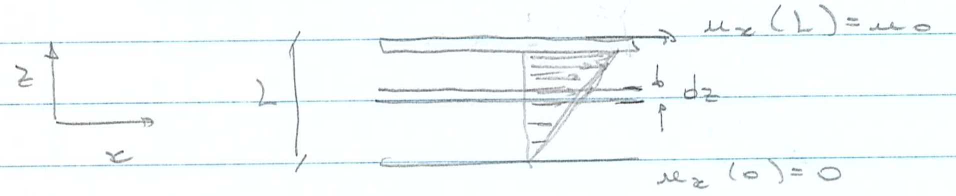
$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \vec{u} \cdot \vec{\nabla} \rho = \underbrace{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u})}_{0} - \rho \vec{\nabla} \cdot \vec{u} = 0$$

$\Rightarrow \vec{\nabla} \cdot \vec{u} = 0$!

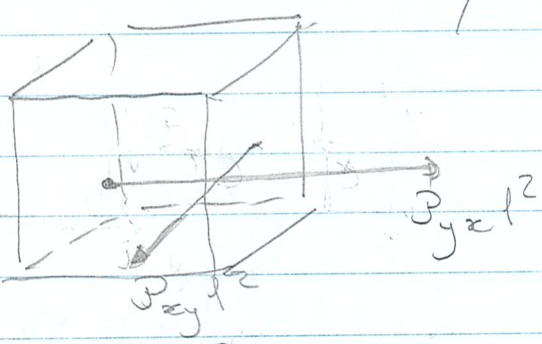
$$\Rightarrow \frac{D\vec{g}}{Dt} = \rho \frac{D\vec{u}}{Dt} = -\vec{\nabla} P \quad ; \text{ equação de Euler}$$

P - pressão

Efeitos de viscosidade:



viscosidade: $\quad -P_{xz} = \eta \dfrac{du_x(z)}{dz}$



$$(P_{xy} l^2 - P_{yx} l^2) l = \tau_z$$

mas $\tau_z = I \dfrac{d\omega_z}{dt} = (P_{xy} - P_{yx}) l^3$

$$I \propto M l^2 = \rho l^5 \Rightarrow \frac{d\omega_z}{dt} = \frac{\tau_z}{I} \propto \frac{(P_{xy} - P_{yx})}{l^2} \to \infty$$

a menos que $P_{xy} = P_{yx}$.

Em geral

$$\rho \frac{D u_\alpha}{Dt} = -\partial_\beta P_{\alpha\beta} + f_\alpha$$

f_α → força p/volume.

Densidades de corrente:

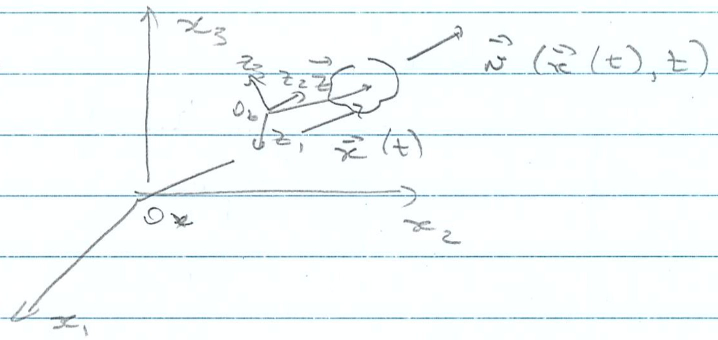
densidade de massa: $\rho = \rho'$

" " momento linear: $\vec{g} = \rho \vec{v}$

" " energia $\epsilon = \epsilon' + \frac{1}{2}\rho v^2$

$R(-\vec{v})$ → laboratório em relação ao fluido em repouso.

Dedução geral:



$\vec{z} = \vec{x}(0)$ $\qquad$ $\vec{x}(t) = \vec{x}(\vec{z}, t)$

Densidade $(\vec{x}, t)$ em movimento

em $t = 0$ $\qquad\qquad$ em t

O_x : $\rho_A(\vec{x}(0), 0)$ $\qquad$ O_x : $\rho_A(\vec{x}(t), t)$

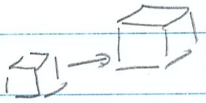
O_z : $\rho'_A(\vec{z}, 0)$ $\qquad$ O_z : $\rho'_A(\vec{z}, t)$

$\rho'_A(\vec{z}, 0) = \rho_A(\vec{x}(0), 0)$ e $\rho'_A(\vec{z}, t) = \rho_A(\vec{x}, t)$

$$\frac{d\rho_A}{dt} = \left.\frac{\partial \rho'_A}{\partial t}\right|_{\vec{z}} = \left.\frac{\partial \rho_A}{\partial t}\right|_{\vec{x}} + \left.\frac{\partial \rho_A}{\partial \vec{x}}\right|_t \cdot \frac{d\vec{x}}{dt}$$

$$\Rightarrow \frac{d\rho_A}{dt} = \left.\frac{\partial \rho'_A}{\partial t}\right|_{z} \quad \left(\neq \left.\frac{\partial \rho_A}{\partial t}\right|_{\vec{x}}\right)$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}_x = \frac{D}{Dt}$$



$\rightarrow dV_t \qquad \vec{x}(0) = \vec{z}$

$\hookrightarrow dV_0$

$$dV_t = J\begin{pmatrix} x_1 \, x_2 \, x_3 \\ z_1 \, z_2 \, z_3 \end{pmatrix} dV_0$$

$dV_0 = d\vec{z} = d\vec{x}(0) \quad \text{e} \quad dV_t = d\vec{x}(t)$

$$J\begin{pmatrix} x_1 \, x_2 \, x_3 \\ z_1 \, z_2 \, z_3 \end{pmatrix} = \det \begin{vmatrix} \partial x_1/\partial z_1 & \partial x_1/\partial z_2 & \partial x_1/\partial z_3 \\ \partial x_2/\partial z_1 & \partial x_2/\partial z_2 & \partial x_2/\partial z_3 \\ \partial x_3/\partial z_1 & \partial x_3/\partial z_2 & \partial x_3/\partial z_3 \end{vmatrix}$$

$$\frac{dJ}{dt} = ? \qquad \frac{d}{dt}\frac{\partial x_i}{\partial z_j} = \frac{\partial}{\partial z_j}\frac{dx_i}{dt} = \frac{\partial}{\partial z_j} v_i$$

$$\frac{dJ}{dt} = \det \begin{vmatrix} \partial v_1/\partial z_1 & \partial v_1/\partial z_2 & \partial v_1/\partial z_3 \\ --- & --- & --- \\ --- & --- & --- \end{vmatrix}$$

$$+ \det \begin{vmatrix} --- & --- & --- \\ \partial v_2/\partial z_1 & \partial v_2/\partial z_2 & \partial v_2/\partial z_3 \\ --- & --- & --- \end{vmatrix}$$

$$+ \det \begin{vmatrix} --- & --- & --- \\ --- & --- & --- \\ \partial v_3/\partial z_1 & \partial v_3/\partial z_2 & \partial v_3/\partial z_3 \end{vmatrix}$$

Usando que $\dfrac{\partial v_i}{\partial z_j} = \sum_k \dfrac{\partial v_i}{\partial x_k}\dfrac{\partial x_k}{\partial z_j}$

$$\frac{dJ}{dt} = \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right) J = \vec{\nabla}_x \cdot \vec{v}\, J$$

$$A(t) = \int_{V(t)} \rho_A(\vec{x},t)\, dV_t$$

$$\frac{dA(t)}{dt} = \frac{d}{dt}\int_{V(t)} \rho_A(\vec{x},t)\, dV_t$$

$$= \frac{d}{dt}\int_{V_0} \rho_A(\vec{x},t)\, J\, dV_0 = \int_{V_0}\left(\frac{d\rho_A(\vec{x},t)}{dt} J + \rho_A \frac{dJ}{dt}\right) dV_0$$

$$= \int_{V_0}\left(\frac{d\rho_A}{dt} + \rho_A \vec{\nabla}_x \cdot \vec{v}\right) J\, dV_0 = \int_{V(t)}\left(\frac{d\rho_A}{dt} + \rho_A \vec{\nabla}_x \cdot \vec{v}\right) dV_t$$

$$\frac{dA}{dt} = \int_{V(t)}\left(\frac{d\rho_A}{dt} + \rho_A \vec{\nabla}_x \cdot \vec{v}\right) dV_t = \int_{V(t)} \sigma_A\, dV_t - \int_{S(t)} \vec{J}_A \cdot d\vec{S}_t$$

$$\Rightarrow \frac{d\rho_A}{dt} = -\rho_A \vec{\nabla} \cdot \vec{v} + \sigma_A - \vec{\nabla} \cdot \vec{J}_A$$

mas $\frac{d\rho_A}{dt} = \frac{D\rho_A}{Dt} = \frac{\partial \rho_A}{\partial t} + \vec{v} \cdot \vec{\nabla}\rho_A$

$$\Rightarrow \boxed{\frac{\partial \rho_A}{\partial t} = -\vec{\nabla} \cdot \left(\rho_A \vec{v} + \vec{J}_A\right) + \sigma_A}$$

i) densidade de massa (sem fontes)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \qquad \Rightarrow \vec{J}_M = \rho \vec{v}$$

ii) Densidade do momento linear

$$\rho \frac{dv_\alpha}{dt} = -\partial_\beta P_{\alpha\beta} + f_\alpha$$

Em geral: $\rho \frac{da}{dt} = \rho \frac{Da}{Dt} = \rho \frac{\partial a}{\partial t} + \rho \vec{v} \cdot \vec{\nabla} a$

$$= \frac{\partial(\rho a)}{\partial t} - a \frac{\partial \rho}{\partial t} + \rho \vec{v} \cdot \vec{\nabla} a =$$

$$= \frac{\partial(\rho a)}{\partial t} - a \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho a \vec{v}) - a \vec{\nabla} \cdot (\rho \vec{v})$$

$$= \frac{\partial(\rho a)}{\partial t} + \vec{\nabla} \cdot (\rho a \vec{v}) \; ; \forall a \Rightarrow \boxed{\rho \frac{da}{dt} = \frac{\partial(\rho a)}{\partial t} + \vec{\nabla} \cdot (\rho a \vec{v})}$$

$$\Rightarrow \rho \frac{dv_\alpha}{dt} = \frac{\partial(\rho v_\alpha)}{\partial t} + \partial_\beta \rho v_\alpha v_\beta = -\partial_\beta P_{\alpha\beta} + f_\alpha$$

$$\Rightarrow \frac{\partial g_\alpha}{\partial t} = -\partial_\beta (P_{\alpha\beta} + \rho v_\alpha v_\beta) + f_\alpha \Rightarrow \boxed{\frac{\partial g_\alpha}{\partial t} + \partial_\beta T_{\alpha\beta} = f_\alpha}$$

$$\Rightarrow \boxed{T_{\alpha\beta} = P_{\alpha\beta} + \rho v_\alpha v_\beta}$$

$f_\alpha = 0 \Rightarrow$ conservação do momento linear!

iii) Densidade de energia:

ϵ - densidade de energia u - densidade de energia interna

$\epsilon = u + \frac{1}{2} \rho v^2$ $\epsilon = \rho e$ e $u = \rho e_u$; $\left.\begin{matrix} e \\ e_u \end{matrix}\right\}$ energias específicas

$$\rho \frac{d}{dt}\left(\frac{v^2}{2}\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \partial_\beta \left(\frac{1}{2} \rho v^2 v_\beta\right)$$

mas $\rho v_\alpha \frac{dv_\alpha}{dt} = -v_\alpha \partial_\beta P_{\alpha\beta} + f_\alpha v_\alpha = \rho \frac{d}{dt}\left(\frac{1}{2} v^2\right)$

$$\Rightarrow \frac{\partial}{\partial t}\left(\rho \frac{v^2}{2}\right) + \partial_\beta \left(\frac{1}{2} \rho v^2 v_\beta\right) = \underbrace{-v_\alpha \partial_\beta P_{\alpha\beta}}_{-\partial_\beta(v_\alpha P_{\alpha\beta}) + P_{\alpha\beta} \partial_\beta v_\alpha} + f_\alpha v_\alpha$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\partial_\beta\left[\frac{1}{2}\rho v^2 v_\beta + v_\alpha P_{\alpha\beta}\right] + \partial_\beta v_\alpha P_{\alpha\beta} + f_\alpha v_\alpha$$

Mas $\frac{\partial \epsilon}{\partial t} = \frac{\partial u}{\partial t} + \frac{\partial}{\partial t}\left(\frac{\rho v^2}{2}\right)$ com $u = \rho e_u$. Como

$\rho \frac{de_u}{dt} = \underbrace{\rho \frac{dq}{dt}}_{-\vec{\nabla}\cdot\vec{J}_Q} - \rho\frac{dw}{dt}$ temos $\frac{\partial u}{\partial t} + \vec{\nabla}\cdot(u\vec{v}) = -\vec{\nabla}\cdot\vec{J}_Q - \rho\frac{dw}{dt}$

$$\Rightarrow \frac{\partial \epsilon}{\partial t} = -\partial_\beta\Big(J_\beta^Q + \underbrace{u v_\beta + \frac{1}{2}\rho v^2 v_\beta}_{\epsilon v_\beta} + v_\alpha P_{\alpha\beta}\Big) + \partial_\beta v_\alpha P_{\alpha\beta} - \rho\frac{dw}{dt}$$

no caso de $f = 0$. Usando que $v = \rho^{-1}$ pode-se mostrar que

$\rho\frac{dv}{dt} = \vec{\nabla}\cdot\vec{v} \Rightarrow$ se $P_{\alpha\beta} = P\delta_{\alpha\beta} + \Pi_{\alpha\beta}$ temos $\underbrace{\rho\left(P\frac{dv}{dt} - \frac{dw}{dt}\right)}_{0} + \partial_\beta v_\alpha \Pi_{\alpha\beta}$

Coeficientes de Transporte e a eq. de Navier-Stokes

$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{g} = 0$ $\qquad \vec{g} = \rho\vec{v} = \vec{J}_M$

$\frac{\partial g_\alpha}{\partial t} + \partial_\beta T_{\alpha\beta} = 0$ $\qquad T_{\alpha\beta} = P_{\alpha\beta} + \rho v_\alpha v_\beta$

$\frac{\partial \epsilon}{\partial t} + \partial_\beta J_\beta^E = 0$ $\qquad J_\beta^{(E)} = J_\beta^{(Q)} + P_{\alpha\beta} v_\alpha + \epsilon v_\beta$

$\vec{J}_M^{(eq)} = \vec{J}_M$; $T_{\alpha\beta}^{(eq)} = P\delta_{\alpha\beta} + \rho v_\alpha v_\beta$

$\vec{J}_E^{(eq)} = (\epsilon + P)\vec{v}$

$L_{ij} \to$ matriz 5×5 (simétrica) $\Rightarrow$ 15 elementos

como $\vec{J}^M = \vec{J}_M^{(eq)} \Rightarrow L_{iM} = L_{Mi} = 0 \Rightarrow$ 12 elementos

$\Rightarrow$ afinidades possíveis : $\partial_\beta\left(\frac{1}{T}\right)$ e $\partial_\beta\left(\frac{v_\alpha}{T}\right)$

No referencial onde $\vec{v}=0 \Rightarrow \frac{1}{T} \quad \partial_\beta v_\alpha$

$\vec{J}'_{(E)}$ $\propto$ vetor $\Rightarrow \vec{\nabla}\left(\frac{1}{T}\right)$ ($\vec{\nabla}\times\vec{v}$ impossível pois é pseudo-vetor)

diferença $P_{\alpha\beta} - P\delta_{\alpha\beta}$ tensor simétrico

$\Rightarrow$ proporcional a $\partial_\alpha v_\beta + \partial_\beta v_\alpha$ ou $\delta_{\alpha\beta}(\partial_\gamma v_\gamma)$
$= \delta_{\alpha\beta}(\vec{\nabla}\cdot\vec{v})$

Combinação: $\Delta_{\alpha\beta} = \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \frac{1}{3}\delta_{\alpha\beta}(\partial_\gamma v_\gamma)$

$$\Rightarrow \begin{cases} \vec{J}'_{(E)} = L_{EE}\,\vec{\nabla}(1/T) \\ P_{\alpha\beta} - P\delta_{\alpha\beta} = -\zeta\,\delta_{\alpha\beta}(\partial_\gamma v_\gamma) - 2\eta\,\Delta_{\alpha\beta} \end{cases}$$

Assumindo $v_x \neq 0$ e $v_x = v_x(z) \Rightarrow \vec{\nabla}\cdot\vec{v} = 0$

$$\Delta_{xz} = \frac{1}{2}\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) = \frac{1}{2}\frac{dv_x}{dz}$$

$$\Rightarrow P_{xz} = -\eta\,\frac{dv_x}{dz}$$

$$P_{\alpha\beta} = P\delta_{\alpha\beta} - \zeta\,\delta_{\alpha\beta}(\vec{\nabla}\cdot\vec{v}) - \eta(\partial_\alpha v_\beta + \partial_\beta v_\alpha) + \frac{2}{3}\eta\,\delta_{\alpha\beta}(\vec{\nabla}\cdot\vec{v})$$

$$\partial_\beta P_{\alpha\beta} = \partial_\alpha P - \zeta\,\partial_\alpha(\vec{\nabla}\cdot\vec{v}) - \eta(\partial^2_\beta v_\alpha) - \eta\,\partial_\alpha(\vec{\nabla}\cdot\vec{v}) + \frac{2}{3}\eta\,\partial_\alpha(\vec{\nabla}\cdot\vec{v})$$

$$\Rightarrow \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} + \frac{\vec{\nabla}P}{\rho} = \frac{\eta}{\rho}\nabla^2\vec{v} + \frac{1}{\rho}\left(\frac{\eta}{3}+\zeta\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{v})$$

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$$\eta = \zeta = 0 \Rightarrow \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla} P}{\rho} \quad \text{(Euler)}$$

Se $\vec{\nabla} P = 0$, $\vec{\nabla}\cdot\vec{v} = 0$ e sem convecção

$$\frac{\partial \vec{v}}{\partial t} = \frac{\eta}{\rho} \nabla^2 \vec{v} \quad \text{(difusão na velocidade)}$$

Produção de entropia:

$$\vec{J}_s{}' = \sum_i \gamma_i \vec{J}_i{}' = \frac{1}{T} \vec{J}_E{}'$$

Prob. (6.5.1) + Invariância de Galileo

$$\vec{J}_s = \vec{J}_s{}' + s\vec{v} = \frac{1}{T}\vec{J}_E{}' + s\vec{v}$$

$$\Rightarrow \frac{\partial s}{\partial t} + \vec{\nabla}\cdot\vec{J}_s = \vec{\nabla}\left(\frac{1}{T}\right)\vec{J}_E{}' + \sum_{\alpha\beta}\left(-\frac{1}{T}\partial_\alpha v_\beta\right)(P_{\alpha\beta} - \delta_{\alpha\beta} P)$$

$$= \kappa T^2 \left(\vec{\nabla}\frac{1}{T}\right)^2 + \frac{\zeta}{T}\left(\vec{\nabla}\cdot\vec{v}\right)^2 + \frac{2\eta}{T}\sum_{\alpha\beta}\left(\Lambda_{\alpha\beta}\right)^2 \geq 0$$

$\Rightarrow \kappa, \eta$ e $\zeta > 0$.