2.3) Equação de Boltzmann.

2.3.1) Termo de colisão.

Condição do caos molecular

$$f^{(2)}(\vec{r}_1,\vec{p}_1;\vec{r}_2,\vec{p}_2;t) = f(\vec{r}_1,\vec{p}_1,t)\, f(\vec{r}_2,\vec{p}_2,t)$$

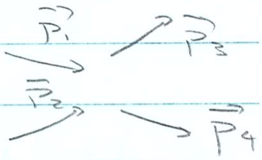
Variável de interesse: $\hat{f}(\vec{r},\vec{p},t) = \sum_{j=1}^{N} \delta(\vec{r}-\vec{r}_j(t))\,\delta(\vec{p}-\vec{p}_j(t))$

$$f(\vec{r},\vec{p},t) = \langle \hat{f}(\vec{r},\vec{p},t)\rangle = \left\langle \sum_{j=1}^{N} \delta(\vec{r}-\vec{r}_j(t))\,\delta(\vec{p}-\vec{p}_j(t))\right\rangle$$

Chamando $f(\vec{r},\vec{p}_1,t) = f_1$ Temos:

$$\frac{\partial f_1}{\partial t} + \vec{v}_1\cdot\vec{\nabla}_r f_1 + \vec{F}(\vec{r})\cdot\vec{\nabla}_{p_1} f_1 = \mathcal{C}[f_1]$$

Ref. do C.M. e colisão elástica.



$$\vec{P} = \vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4$$

$$\vec{p}_3{}' = \vec{p}_3 - \frac{\vec{P}}{2} = \frac{1}{2}(\vec{p}_3 - \vec{p}_4)$$

$$\vec{p}_4{}' = \vec{p}_4 - \frac{\vec{P}}{2} = -\frac{1}{2}(\vec{p}_3 - \vec{p}_4) = -\vec{p}_3{}'$$

e relações idênticas $(3) \leftrightarrow (1)$ e $(2) \leftrightarrow (4)$

Energias: $$\varepsilon_3 = \frac{p_3^2}{2m} = \frac{1}{2m}\left(\vec{p}_3{}' + \frac{\vec{P}}{2}\right)^2 = \frac{1}{2m}p_3'^2 + \frac{1}{2m}\vec{p}_3{}'\cdot\vec{P} + \frac{P^2}{8m}$$

$$\varepsilon_4 = \frac{p_4^2}{2m} = \frac{1}{2m}\left(-\vec{p}_3{}' + \frac{\vec{P}}{2}\right)^2 = \frac{1}{2m}p_3'^2 - \frac{1}{2m}\vec{p}_3{}'\cdot\vec{P} + \frac{P^2}{8m}$$

$$\Rightarrow \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) = \delta\left(\frac{p_1'^2}{m} - \frac{p_3'^2}{m}\right)$$

Vamos definir $\overline{w}(\vec{p}_1, \vec{p}_2; \vec{p}_3, \vec{p}_4) = \frac{4}{m^2}\sigma(|\vec{v}_1 - \vec{v}_2|, \Omega')$

$$\frac{dN}{dt} = \int d^3p_3 d^3p_4 \, \overline{w} \, \delta(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)\, \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$$

$$= \int d^3p_3' \, \overline{w} \, \delta\left(\frac{p_1'^2}{m} - \frac{p_3'^2}{m}\right)$$

$$= \int dp_3' (p_3')^2 d\Omega' \, \overline{w} \, \delta\left(\frac{p_1'^2}{m} - \frac{p_3'^2}{m}\right)$$

$$= \frac{m}{2} p_1' \int d\Omega' \overline{w} = \frac{2p_1'}{m} \int d\Omega' \sigma(|\vec{v}_1 - \vec{v}_2|, \Omega')$$

Mas $\vec{p}_1' = \frac{1}{2}(\vec{p}_1 - \vec{p}_2) \quad \vec{p}_2' = -\vec{p}_1'$

$$\Rightarrow \frac{2p_1'}{m} = \frac{1}{m}|\vec{p}_1 - \vec{p}_2| = |\vec{v}_1 - \vec{v}_2|$$

$$\Rightarrow \frac{dN}{dt} = |\vec{v}_1 - \vec{v}_2| \int d\Omega' \sigma(|\vec{v}_1 - \vec{v}_2|, \Omega') = |\vec{v}_1 - \vec{v}_2| \, \sigma_{TOT}(|\vec{v}_1 - \vec{v}_2|)$$

$\frac{dN}{dt}$ = # colisões/tempo - partícula incidente-partícula alvo

$$\Rightarrow w(\vec{p}_1, \vec{p}_2 \to \vec{p}_3, \vec{p}_4)\, d^3p_3 d^3p_4 = \overline{w}\, \delta(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \, * \\ * \, \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\, d^3p_3 d^3p_4$$

= # colisões/tempo - part. inc - part. alvo em $d^3p_3 d^3p_4$

$\Rightarrow \mathcal{C}_-[f_1]\, d^3r\, d^3p_1$ = # partículas saindo de $d^3r\, d^3p_1$

$$= f(\vec{r}, \vec{p}_1, t)\, d^3r\, d^3p_1 \int d^3p_2\, d^3p_3\, d^3p_4\, f(\vec{r}, \vec{p}_2, t)\, w(\vec{p}_1, \vec{p}_2 \to \vec{p}_3, \vec{p}_4)$$

Analogamente:

$\mathcal{C}_+[f_1]\, d^3r\, d^3p_1$ = # partículas entrando em $d^3r\, d^3p_1$

$$= d^3r\, d^3p_1 \int d^3p_2\, d^3p_3\, d^3p_4\, f(\vec{r},\vec{p}_3,t)\, f(\vec{r},\vec{p}_4,t)\, w(\vec{p}_3\vec{p}_4 \to \vec{p}_1\, \vec{p}_2)$$

Invariância de w

i) rotação $w(R\vec{p}_1, R\vec{p}_2 \to R\vec{p}_3, R\vec{p}_4) = w(\vec{p}_1\vec{p}_2 \to \vec{p}_3\,\vec{p}_4)$

ii) paridade $w(-\vec{p}_1, -\vec{p}_2 \to -\vec{p}_3, -\vec{p}_4) = w(\vec{p}_1\,\vec{p}_2 \to \vec{p}_3\,\vec{p}_4)$

iii) reversão temporal: $w(-\vec{p}_3, -\vec{p}_4 \to -\vec{p}_1, -\vec{p}_2) = w(\vec{p}_1\,\vec{p}_2 \to \vec{p}_3\,\vec{p}_4)$

$\Rightarrow$ (ii) + (iii) $\quad w(\vec{p}_1, \vec{p}_2 \to \vec{p}_3, \vec{p}_4) = w(\vec{p}_3, \vec{p}_4 \to \vec{p}_1, \vec{p}_2)$

$\Rightarrow$ combinando $\mathcal{C}_-$ e $\mathcal{C}_+$ temos:

$$\mathcal{C}[f_1] = \int \prod_{i=2}^{4} d^3p_i\; w(\vec{p}_1\vec{p}_2 \to \vec{p}_3\vec{p}_4)\,[f_3 f_4 - f_1 f_2]$$

onde $f_i = f(\vec{r}, \vec{p}_i, t)$. Assim,

$$\frac{\partial f_1}{\partial t} + \vec{v}_1 \cdot \vec{\nabla}_r f_1 + \vec{F}(\vec{r}) \cdot \vec{\nabla}_p f_1 = \int \prod_{i=2}^{4} d^3p_i\; w(\vec{p}_1\vec{p}_2 \to \vec{p}_3\vec{p}_4)\,[f_3 f_4 - f_1 f_2]$$

$$= \int d\Omega' \int d^3p_2\, \sigma(|\vec{v}_1 - \vec{v}_2|, \Omega')\, |\vec{v}_1 - \vec{v}_2|\,[f_3 f_4 - f_1 f_2]$$

Se $f_2 = f_4 = n_d\, \delta(\vec{p}) \Rightarrow$ Boltzmann-Lorentz!

Eq. Boltzmann quebra invariância temporal porque $f(\vec{r}, -\vec{p}, -t)$ ñ obedece B.E. pq. termo de arraste troca sinal mas não o de colisão.

2.3.2) Leis de conservação.

massa, energia + 3 componentes do momento.

$$\vec{p}_1 + \vec{p}_2 \to \vec{p}_3 + \vec{p}_4$$

Seja $\chi(\vec{p})$ conservada $\Rightarrow$ $\chi_1 + \chi_2 = \chi_3 + \chi_4$ $\quad (\chi_i = \chi(\vec{p}_i))$

Devemos demonstrar que $I[\chi] = \int d^3p_1\, \chi(\vec{p}_1)\, \mathcal{C}[f_1] = 0$

$$I[\chi] = \int \prod_{i=1}^{4} d^3p_i\, \chi_1\, w(12 \to 34)\, [f_3 f_4 - f_1 f_2]$$

$w(12 \to 34) = w(21 \to 34)$ partículas idênticas

e $\vec{p}_1 \leftrightarrow \vec{p}_2$

$$I[\chi] = \int \prod_{i=1}^{4} d^3p_i\, \chi_2\, w(12 \to 34)\, [f_3 f_4 - f_1 f_2]$$

Outra expressão $(12) \leftrightarrow (34)$ e colisão inversa

$w(34 \to 12) = w(12 \to 34)$

$$I[\chi] = -\int \prod_{i=1}^{4} d^3p_i\, \chi_3\, w(12 \to 34)\, [f_4 f_3 - f_1 f_2]$$

e finalmente $(3) \leftrightarrow (4)$ obtem-se

$$I[\chi] = \frac{1}{4} \int \prod_{i=1}^{4} d^3p_i\, [\chi_1 + \chi_2 - \chi_3 - \chi_4]\, w(12 \to 34)\, [f_3 f_4 - f_1 f_2]$$

$= 0$ porque χ é conservada!

Nada muda se $\chi = \chi(\vec{r}, \vec{p})$

$\Rightarrow$ multiplicando B. Eq. por $\chi(\vec{r}, \vec{p}_1) = \chi(\vec{r}, \vec{p})$

$(\vec{p}_1 \to \vec{p})$

$$\int d^3p\, \chi(\vec{r},\vec{p})\left(\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla} f + \vec{F}(\vec{r})\,\vec{\nabla}_p f\right) = 0$$

Vamos usar $\int d^3p\, \chi\, v_\alpha \partial_\alpha f = \partial_\alpha \int d^3p\, (\chi\, v_\alpha f) - \int d^3p\, f\, v_\alpha \partial_\alpha \chi$

e $$\int d^3p\, \chi F_\alpha \partial_{p_\alpha} f = \underbrace{\int d^3p\, \partial_{p_\alpha}(\chi F_\alpha f)}_{\int \vec{\nabla}\cdot(\,)\,d^3p = 0} - \int d^3p\, (\partial_{p_\alpha}\chi) F_\alpha f \underbrace{- \int d^3p\, \chi\, (\partial_{p_\alpha} F_\alpha) f}_{F_\alpha \neq F_\alpha(p_\alpha)} \Rightarrow 0$$

$$= -\int d^3p\, (\partial_{p_\alpha}\chi) F_\alpha f$$

Mas, $\langle \chi \rangle = \frac{1}{n}\int d^3p\, f\chi$ e $\vec{J}_\chi = \int d^3p\, \vec{v}\, \chi f = n\langle \vec{v}\chi\rangle$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}(n\langle\chi\rangle) + \partial_\alpha(n\langle v_\alpha \chi\rangle) = n\langle v_\alpha \partial_\alpha \chi\rangle + n\langle F_\alpha \partial_{p_\alpha}\chi\rangle}$$

Sem fontes: $\boxed{\partial_t \rho_\chi + \vec{\nabla}\cdot\vec{J}_\chi = 0}$

Podemos escrever $\vec{v}$ como $\vec{v} = \vec{u} + \vec{w}$

onde $\langle\vec{v}\rangle = \vec{u} \Rightarrow \langle\vec{v}\rangle = \vec{u} + \langle\vec{w}\rangle$ e $\langle\vec{w}\rangle = 0$

$\Rightarrow \vec{w}$ é medida no ref. em repouso $\Rightarrow$

devida a flutuações térmicas.

Se $\chi = m$, $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{g} = 0$ onde $\rho = nm$

e $\vec{g} = \rho\langle\vec{v}\rangle = \rho\vec{u}$

$\chi = m v_\beta \Rightarrow \partial_{p_\alpha} \chi = \delta_{\alpha\beta}$

$$\frac{\partial}{\partial t}(\rho \langle v_\beta \rangle) + \partial_\alpha(\rho \langle v_\alpha v_\beta \rangle) = n F_\beta$$

$n F_\beta = f_\beta \Rightarrow$ densidade de força e $T_{\alpha\beta} = \rho \langle v_\alpha v_\beta \rangle$

Se $\vec{v} = \vec{u} + \vec{w}$, $T_{\alpha\beta} = \rho u_\alpha u_\beta + \rho \langle w_\alpha w_\beta \rangle$

$\Rightarrow P_{\alpha\beta} = \rho \langle w_\alpha w_\beta \rangle$ e $P_{\alpha\alpha} = nm \langle w^2 \rangle = \rho \langle w^2 \rangle$

Finalmente, $\chi = m v^2/2$; $\epsilon = \frac{1}{2}\rho v^2$ e $\vec{J}_E = \frac{1}{2}\rho \langle v^2 \vec{v} \rangle$

se $f = 0 \Rightarrow \frac{\partial \epsilon}{\partial t} + \vec{\nabla} \cdot \vec{J}_E = 0$

No referencial em repouso temos $\vec{J}_E' = \vec{J}_q$

Usando $\vec{v} = \vec{u} + \vec{w}$ temos $J^E_\alpha = \epsilon u_\alpha + u_\beta P_{\alpha\beta} + J^{E'}_\alpha$

$\Rightarrow$ Temperatura local $\frac{1}{2} m \langle w^2 \rangle = \frac{3}{2} k_B T(\vec{r}, t)$

$\Rightarrow \rho \langle w^2 \rangle = 3 n k_B T = 3 P$

$\Rightarrow \sum_\alpha P_{\alpha\alpha} - 3P = 0$