Elements of Superconductivity

A. London theory of superconductivity

Superconductivity is a property common to several metals. Below a given transition temperature they present;

i) transport of charge with no measurable resistance

ii) perfect diamagnetism; Meissner effect

Phenomenologically these two effects can be described by the following equations:

London equations

$$\mathbf{E} = \frac{\partial}{\partial t}(\Lambda \mathbf{J}_s)$$
$$\mathbf{B} = -c\nabla \times (\Lambda \mathbf{J}_s)$$

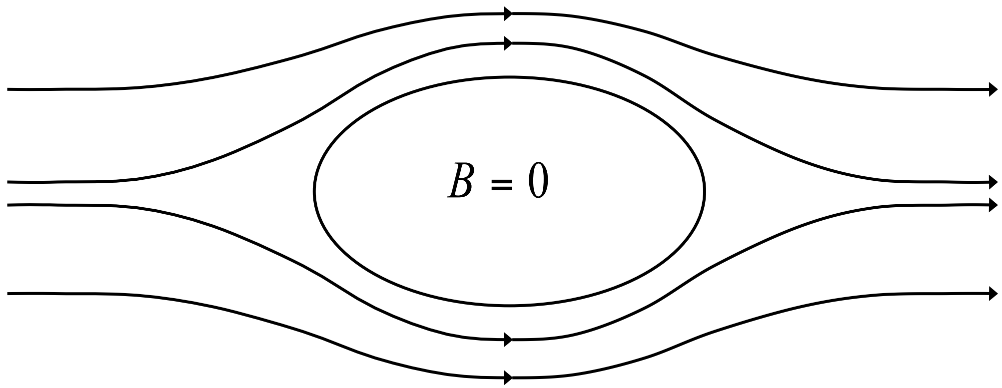
Combined with the Maxwell equation

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}_s$$

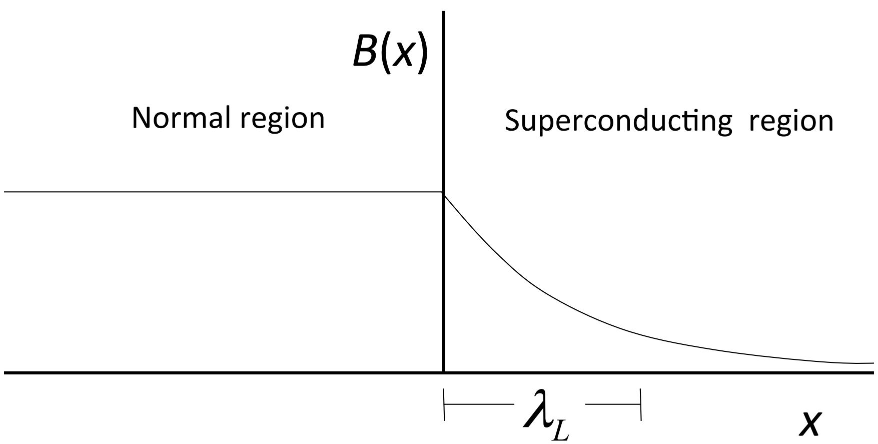
Result in the equation that describes the Meissner effect

$$\nabla^2 \mathbf{B} = \frac{\mathbf{B}}{\lambda_L^2}$$

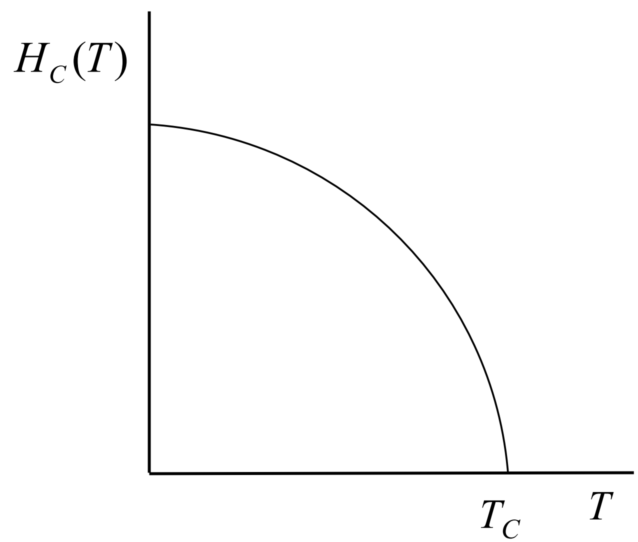
where $\Lambda = 4\pi {\lambda_L}^2/c^2$



$B = 0$



$B(x)$
Normal region
Superconducting region
λ_L
x



$H_C(T)$
T_C
T

If the external scalar potential is zero

$$\mathbf{E} = \frac{\partial}{\partial t}(\Lambda \mathbf{J}_s) = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

This implies that

$$\mathbf{J}_s = -\frac{1}{c\Lambda}(\mathbf{A} - \mathbf{A}_0)$$

Choosing (London gauge)

$$\mathbf{J}_{s\perp} = \frac{1}{c\Lambda}\mathbf{A}_0 = 0$$

London equation in the London gauge

$$\boxed{\mathbf{J}_s = -\frac{1}{c\Lambda}\mathbf{A}}$$

Bloch's theorem for the ground state of the superconductor

$$\langle \mathbf{p} \rangle = 0$$

applied to the system in a magnetic field

$$\mathbf{p} = m\dot{\mathbf{r}} + e\frac{\mathbf{A}}{c}$$

yields

$$\mathbf{J}_s = n_s e \langle \mathbf{v}_s \rangle = -\frac{n_s e^2 \mathbf{A}}{mc} = -\frac{1}{c\Lambda}\mathbf{A}$$

and the London penetration depth is given by

$$\lambda_L = \left(\frac{mc^2}{4\pi n_s e^2} \right)^{1/2}$$

For the ground state wavefunction of the superconductor

$$\mathbf{J}_s = \frac{e\hbar}{2mi}\left[\psi_0^*\nabla\psi_0 - \psi_0\nabla\psi_0^*\right] = \frac{e}{m}\ \mathrm{Re}\ [\psi_0^*\mathbf{p}\psi_0] = 0$$

In an external field it changes to

$$\mathbf{J}_s = \frac{e\hbar}{2mi}\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right] - \frac{e^2\mathbf{A}}{mc}\psi^*\psi$$

If the wave function is rigid $\psi \approx \psi_0$

$$\mathbf{J}_s = -\frac{e^2\mathbf{A}}{mc}\psi^*\psi = -\frac{ne^2\mathbf{A}}{mc}$$

Current - carrying wave function (constant velocity)

$$\Psi_{\mathbf{K}}(\mathbf{r}_1, ..., \mathbf{r}_k, ..., \mathbf{r}_N) = \exp\{i \sum_k \mathbf{K} \cdot \ \mathbf{r}_k\} \, \Psi_0(\mathbf{r}_1, ..., \mathbf{r}_k, ..., \mathbf{r}_N).$$

Generalization to position dependent velocity

$$\Psi(\mathbf{r}_1, ..., \mathbf{r}_k, ..., \mathbf{r}_N) = \exp \ \{i \sum_k \theta(\mathbf{r}_k)\} \ \Psi_0(\mathbf{r}_1, ..., \mathbf{r}_k, ..., \mathbf{r}_N).$$

Number and current densities

$$n(\mathbf{r}) = \sum_k \int d\mathbf{r}_1 ... d\mathbf{r}_k ... d\mathbf{r}_N \delta(\mathbf{r} - \mathbf{r}_k) \Psi^*(\mathbf{r}_1, ..., \mathbf{r}_k, ..., \mathbf{r}_N) \Psi(\mathbf{r}_1, ..., \mathbf{r}_k, ..., \mathbf{r}_N),$$

$$\mathbf{J}(\mathbf{r}) = \sum_k \int d\mathbf{r}_1 ... d\mathbf{r}_k ... d\mathbf{r}_N \frac{e\hbar}{2mi} \left[\Psi^* \nabla_k \Psi - \Psi \nabla_k \Psi^* \right] \delta(\mathbf{r} - \mathbf{r}_k).$$

These imply in a current density with a number density → $$\mathbf{J}(\mathbf{r}) = \frac{e\hbar}{m} n(\mathbf{r}) \nabla \theta$$

$$n(\mathbf{r}) = N \int d\mathbf{r}_2 \ldots d\mathbf{r}_k \ldots d\mathbf{r}_N \Psi^*(\mathbf{r}, \ldots, \mathbf{r}_k, \ldots, \mathbf{r}_N) \Psi(\mathbf{r}, \ldots, \mathbf{r}_k, \ldots, \mathbf{r}_N),$$

Macroscopic occupation of a single particle state or center-of-mass of a translationally invariant system → $$n(\mathbf{r}) = N \psi^*(\mathbf{r}) \psi(\mathbf{r}),$$

Single particle wavefunction → $$\psi(\mathbf{r}) = \sqrt{n(\mathbf{r})} e^{i\theta(\mathbf{r})}.$$

Non-ideal rigidity → $$\mathbf{J}(\mathbf{r}) = e n_s(\mathbf{r}) \mathbf{v}_s(\mathbf{r}) + e n_N(\mathbf{r}) \mathbf{v}_N(\mathbf{r}),$$

More generally, we can define the 1-particle reduced density operator

$$n_1(\mathbf{r}\,;\mathbf{r}') = N \int d\mathbf{r}_2 ... d\mathbf{r}_k ... d\mathbf{r}_N \Psi_0(\mathbf{r}, ..., \mathbf{r}_k, ..., \mathbf{r}_N)\Psi_0^*(\mathbf{r}', ..., \mathbf{r}_k, ..., \mathbf{r}_N)$$

From which $n(\mathbf{r}) = n_1(\mathbf{r}\,;\mathbf{r})$

Given general diagonal 1-body operator $\hat{\mathcal{O}}_1 \equiv \sum_i \mathcal{O}_1(\mathbf{r}_i)$ we can write

$$\langle \hat{\mathcal{O}}_1 \rangle = \mathrm{tr}[\hat{n}_1 \hat{\mathcal{O}}_1] = \int d\mathbf{r}\, n_1(\mathbf{r}\,;\mathbf{r})\mathcal{O}_1(\mathbf{r})$$

For a bosonic ground state given by a product state of single particle wave functions $\psi(\mathbf{r})$

$$n(\mathbf{r}) = N\psi^*(\mathbf{r})\psi(\mathbf{r})$$

Average of the effective inter-electronic interaction

$$\hat{V} = \frac{1}{2} \sum_{i \neq j} V(\mathbf{r}_i - \mathbf{r}_j)$$

$$\langle \hat{V} \rangle = \frac{1}{2} \sum_{i \neq j} \int d\mathbf{r}_1 ... d\mathbf{r}_i ... d\mathbf{r}_j ... d\mathbf{r}_N \, V(\mathbf{r}_i - \mathbf{r}_j) \, |\Psi_0(\mathbf{r}_1, ..., \mathbf{r}_i, ..., \mathbf{r}_j, ..., \mathbf{r}_N)|^2$$

$$= \frac{N(N-1)}{2} \int d\mathbf{x} \, d\mathbf{y} \, d\mathbf{r}_3 ... d\mathbf{r}_N \, V(\mathbf{x} - \mathbf{y}) \, |\Psi_0(\mathbf{x}, \mathbf{y}, \mathbf{r}_3, ..., \mathbf{r}_N)|^2$$

Defining the 2-particle reduced density operator

$$n_2(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}') \equiv N(N-1) \int d\mathbf{r}_3 ... d\mathbf{r}_N \Psi_0(\mathbf{x}, \mathbf{y}, \mathbf{r}_3, ..., \mathbf{r}_N) \Psi_0^*(\mathbf{x}', \mathbf{y}', \mathbf{r}_3, ..., \mathbf{r}_N)$$

We can rewrite $\langle \hat{V} \rangle = \frac{1}{2} \int d\mathbf{x} \, d\mathbf{y} \, n_2(\mathbf{x}, \mathbf{y}\,; \mathbf{x}, \mathbf{y}) \, V(\mathbf{x} - \mathbf{y})$

For a general diagonal 2-body operator

$$\hat{\mathcal{O}}_2 \equiv \frac{1}{2} \sum_{i,j} \mathcal{O}_2(\mathbf{r}_i, \mathbf{r}_j)$$

We have $\quad \langle \hat{\mathcal{O}}_2 \rangle = \frac{1}{2} \mathrm{tr}[\hat{n}_2 \hat{\mathcal{O}}_2] = \frac{1}{2} \int d\mathbf{x}\, d\mathbf{y}\, n_2(\mathbf{x}, \mathbf{y}\,; \mathbf{x}, \mathbf{y}) \mathcal{O}_2(\mathbf{x}, \mathbf{y})$

If the ground state is a properly anti-symmetrized product of single particle wave functions

$$n_2(\mathbf{x}, \mathbf{y}\,; \mathbf{x}, \mathbf{y}) = N(N-1)\, \phi^*(\mathbf{x}, \mathbf{y}) \phi(\mathbf{x}, \mathbf{y})$$

Using the center-of-mass and relative coordinates

$$\mathbf{r} \equiv \frac{1}{2} (\mathbf{x} + \mathbf{y}) \text{ and } \mathbf{u} \equiv \mathbf{x} - \mathbf{y}$$

We have

$$n_1(\mathbf{r}\,;\mathbf{r}') = \frac{1}{N-1}\int d\mathbf{u}\; n_2(\mathbf{r},\mathbf{u}\,;\mathbf{r}',\mathbf{u})$$

Assuming translation invariance $\phi(\mathbf{r},\mathbf{u}) = \psi(\mathbf{r})\chi(\mathbf{u})$

and then $n(\mathbf{r}) \equiv n_1(\mathbf{r}\,;\mathbf{r}) = N\,\psi^*(\mathbf{r})\psi(\mathbf{r})$

as before, but with a new interpretation.

Then $\psi(\mathbf{r}) = \sqrt{n(\mathbf{r})}e^{i\theta(\mathbf{r})}$. Finite temperature effects implies depletion of the condensate (two-fluid model)

↓

$$\mathbf{J}(\mathbf{r}) = en_s(\mathbf{r})\mathbf{v}_s(\mathbf{r}) + en_N(\mathbf{r})\mathbf{v}_N(\mathbf{r}),$$

And also invalidate pure state description → density operators, ODLRO, order parameter etc.

Flux Quantization

Canonical momentum in an external field

$$\hbar \nabla \theta = m\mathbf{v} + e\mathbf{A}/c$$

Integrated along an open path

$$\int_1^2 \left(m\mathbf{v} + \frac{e}{c}\mathbf{A} \right) . d\mathbf{r} = \hbar(\theta_2 - \theta_1)$$

Along a closed path deep into a super-conductor

$$\oint \left(m\mathbf{v} + \frac{e}{c}\mathbf{A} \right) . d\mathbf{r} = \frac{e}{c} \oint \left(c\Lambda \mathbf{J} + \mathbf{A} \right) . d\mathbf{r} = 2\pi n \hbar$$

Multiply connected region

$$\oint \left(\Lambda \mathbf{J} + \mathbf{A} \right) . d\mathbf{r} = -\frac{2\pi n \hbar c}{2\,|e|} \equiv n\phi_0$$

Flux quantization ➡

$$\boxed{\oint \mathbf{A}.d\mathbf{r} = \int \mathbf{B}.d\mathbf{s} = n\phi_0}$$

Josephson Effect

Decoupled superconductors

$$\begin{aligned} i\hbar\dot{\psi}_1 &= E_1\psi_1 \\ i\hbar\dot{\psi}_2 &= E_2\psi_2. \end{aligned}$$

Superconductors coupled by a junction

$$\begin{aligned} i\hbar\dot{\psi}_1 &= E_1\psi_1 + \Delta\psi_2 \\ i\hbar\dot{\psi}_2 &= E_2\psi_2 + \Delta^*\psi_1. \end{aligned}$$

Resulting equations for the phase and number density

$$-\hbar\dot{\theta}_1 = \Delta\sqrt{\frac{n_2}{n_1}}\cos(\theta_2 - \theta_1) + E_1$$

$$\dot{n}_1 = \frac{2\Delta}{\hbar}\sqrt{n_1 n_2}\sin(\theta_2 - \theta_1) = -\dot{n}_2.$$

Josephson relations →

$$\begin{aligned} j &= j_0 \sin\theta \\ \dot{\theta} &= \frac{-2|e|V}{\hbar} \end{aligned}$$

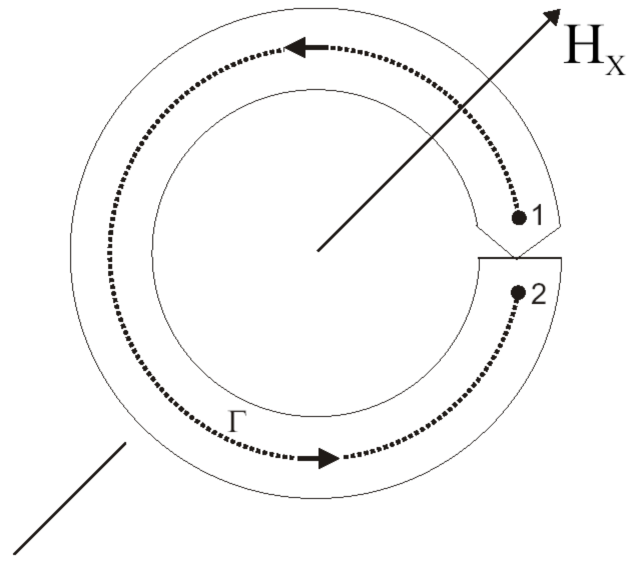
where

$$\theta \equiv \theta_1 - \theta_2, \quad j_0 \equiv \frac{2n_s\Delta}{\hbar} \quad \text{and} \quad \frac{E_2 - E_1}{\hbar} = \frac{-2|e|V}{\hbar}.$$

B. Superconducting devices

Superconducting Quantum Interference Devices (SQUIDs)

Superconducting ring closed by a Josephson junction



Modification of the flux quantization rule

$$\int_1^2 \mathbf{J}.d\mathbf{r} = \frac{n_s e\hbar}{m}\int_1^2 \nabla\theta.d\mathbf{r} - \frac{n_s e\hbar}{m}\frac{2\pi}{\phi_0}\int_1^2 \mathbf{A}.d\mathbf{r}.$$

$$\int_\Gamma \nabla\theta.d\mathbf{r} = \oint \nabla\theta.d\mathbf{r} - \int_2^1 \nabla\theta.d\mathbf{r} \quad \Longrightarrow \quad \int_\Gamma \nabla\theta.d\mathbf{r} = 2\pi n - \Delta\theta$$

Then $\boxed{\phi + \frac{\phi_0}{2\pi}\Delta\theta = n\phi_0.}$ where $\phi = \phi_x + Li$

$$i_s = i_0 \sin\Delta\theta \qquad i_N = \frac{V}{R} \qquad i_c = C\dot{V}$$

and $$i = i_0 \sin\Delta\theta + \frac{V}{R} + C\dot{V}$$

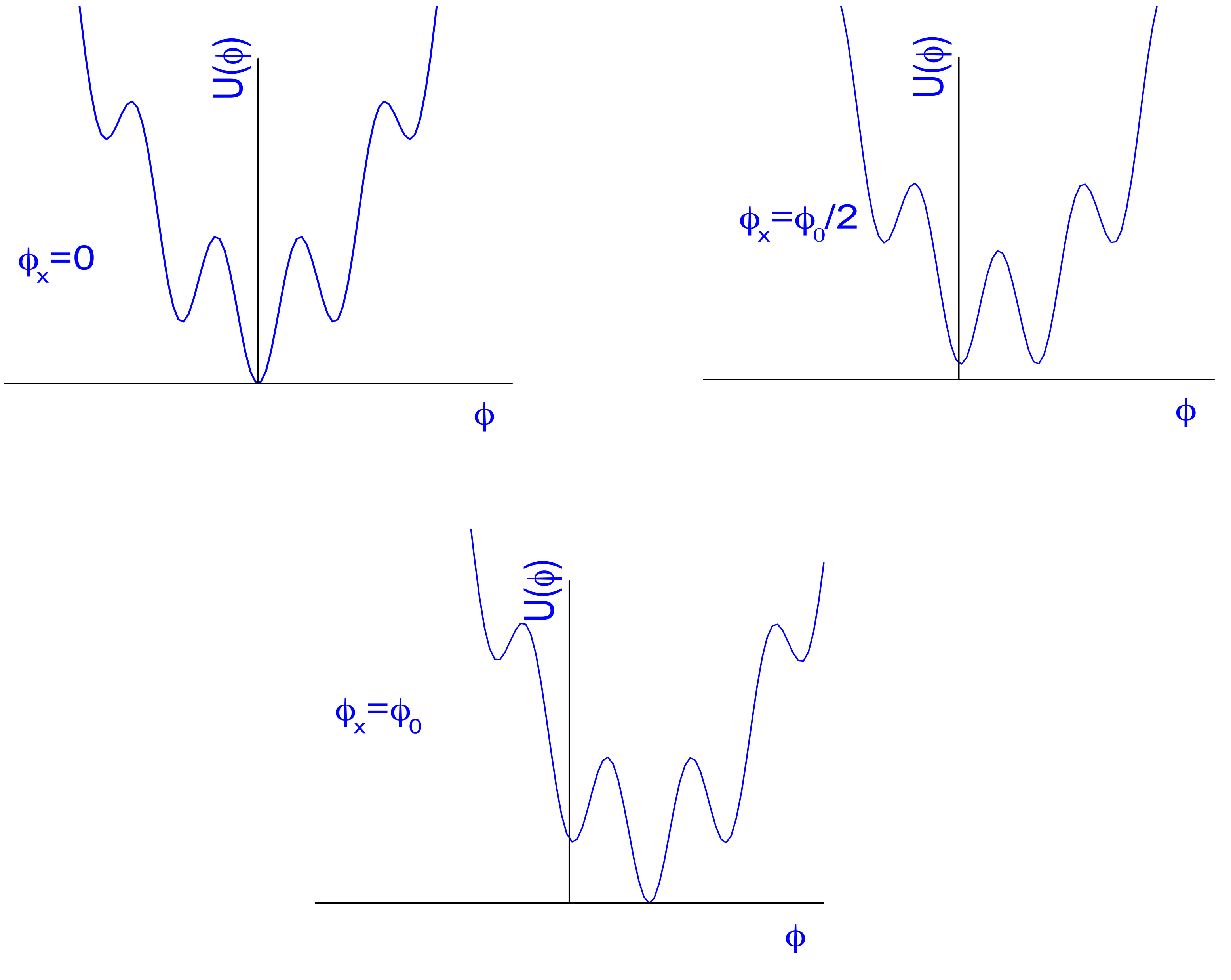
Using that $V = -\dot{\phi}$ $\Rightarrow$ $$\frac{\phi_x - \phi}{L} = i_0 \sin\frac{2\pi\phi}{\phi_0} + \frac{\dot{\phi}}{R} + C\ddot{\phi}$$

Electromagnetic potential energy

$$U(\phi) = \frac{(\phi - \phi_x)^2}{2L} - \frac{\phi_0 i_0}{2\pi}\cos\frac{2\pi\phi}{\phi_0}$$

Equation of motion for the Total flux in the ring

$$C\ddot{\phi} + \frac{\dot{\phi}}{R} + U'(\phi) = 0$$



U(φ)
φx=0
φ
U(φ)
φx=φ0/2
φ
U(φ)
φx=φ0
φ

Current Biased Josephson Junction (CBJJs)

Phase-flux relation

$$\phi = -\frac{\phi_0}{2\pi}\,\Delta\theta \equiv \frac{\phi_0}{2\pi}\,\varphi$$

SQUID ring such that

$$L \to \infty,\ \phi_x \to \infty \text{ but } \phi_x/L = I_x$$

Washboard potential

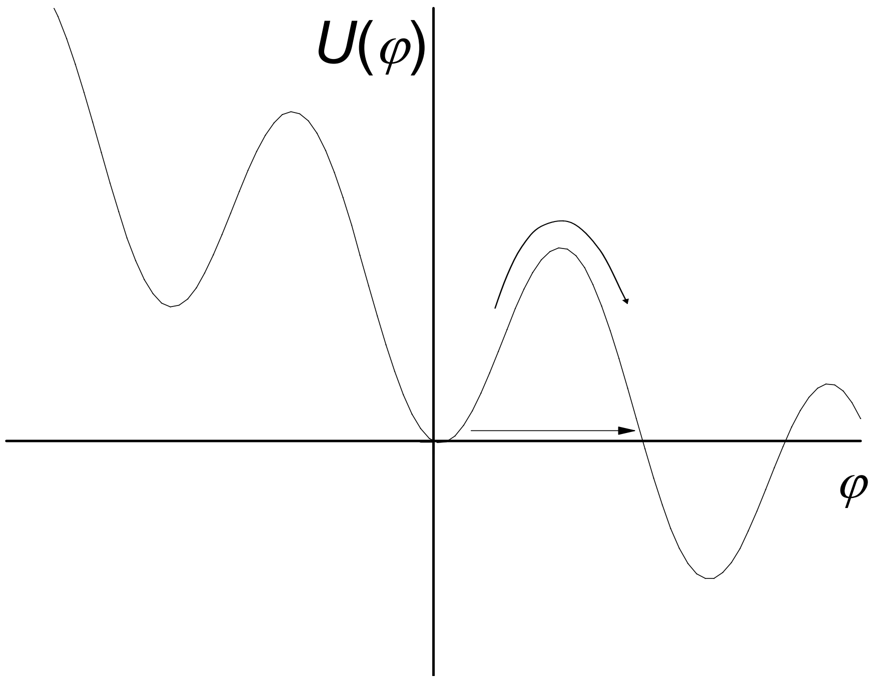
$$\boxed{U(\varphi) = -I_x\,\varphi - i_0\,\cos\varphi}$$

Equation of motion for the phase

$$\boxed{\frac{\phi_0}{2\pi}\,C\,\ddot{\varphi} + \frac{\phi_0}{2\pi R}\,\dot{\varphi} + U'(\varphi) = 0}$$

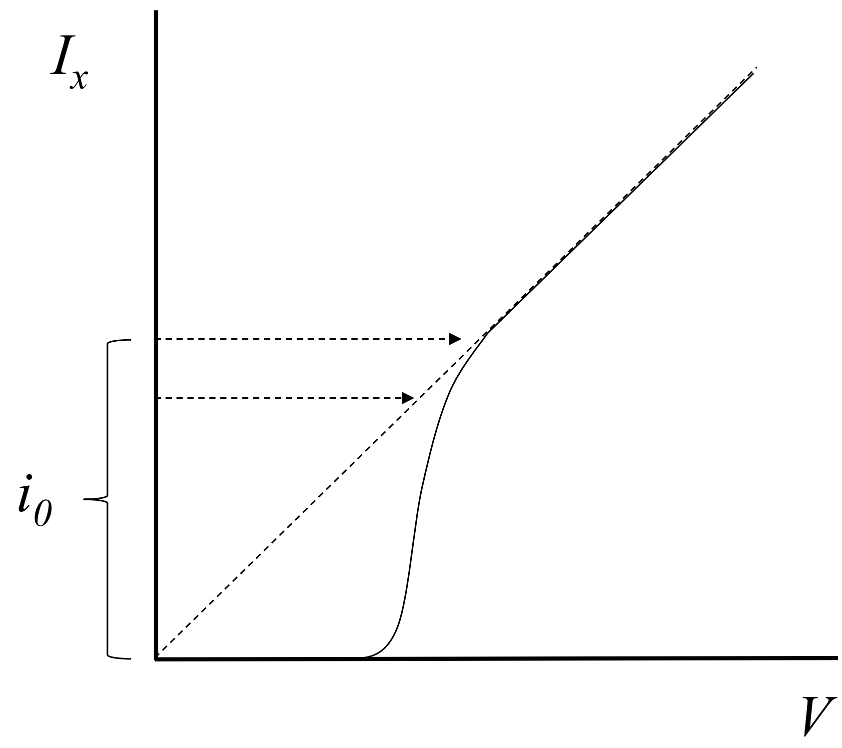
Josephson coupling energy

$$\boxed{E_J \equiv \frac{\phi_0 i_0}{2\pi}}$$



Washboard potential

V x I carachteristic of the CBJJ



Cooper Pair Boxes (CPBs)

Charging energy $E_C = \frac{e^2}{2C}$ If $E_C \gg E_J$

" Nearly free-electron model " for the phase in a periodic potential

$$H_0 = \frac{Q^2}{2C} + U(\varphi) \qquad \text{where} \qquad Q = -i\hbar \frac{\mathrm{d}}{\mathrm{d}(\phi_0 \varphi / 2\pi)}$$

Bloch´s theorem $\psi_{n\,q}(\varphi) = \exp\left\{ i \left(\frac{q}{2e} \right) \varphi \right\} \, u_n(\varphi)$

with $u_n(\varphi + 2\pi) = u_n(\varphi)$

where $q(t) = q_0 + Q_x(t)$ and $Q_x(t) = \int_{t_0}^{t} dt' I_x(t')$

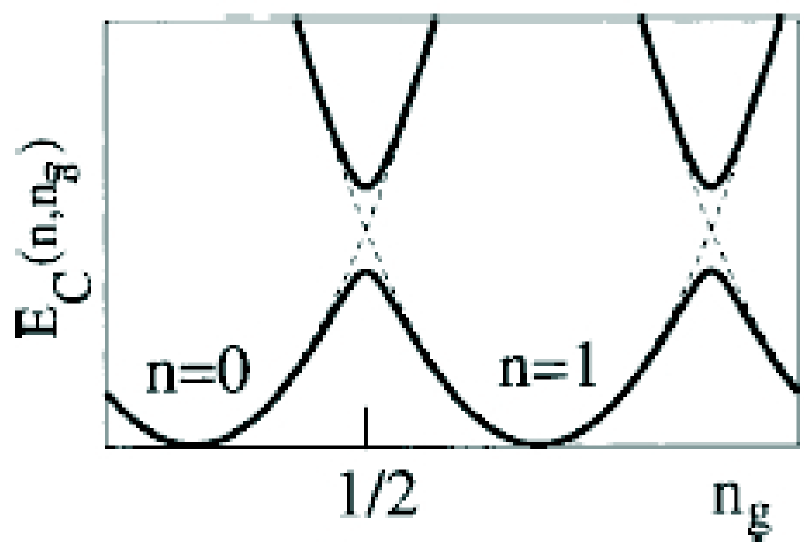
New Schrödinger equation (adiabatic approximation)

$$\mathcal{H}_q u_n(\varphi) = \frac{(Q+q)^2}{2C} u_n(\varphi) + U(\varphi) u_n(\varphi) \;=\; E_n(q) u_n(\varphi)$$

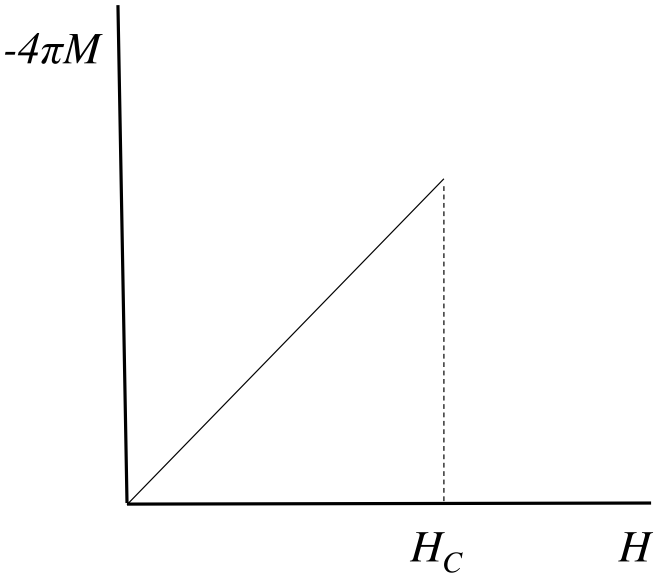
where $$Q = -2ie\frac{\partial}{\partial\varphi}$$

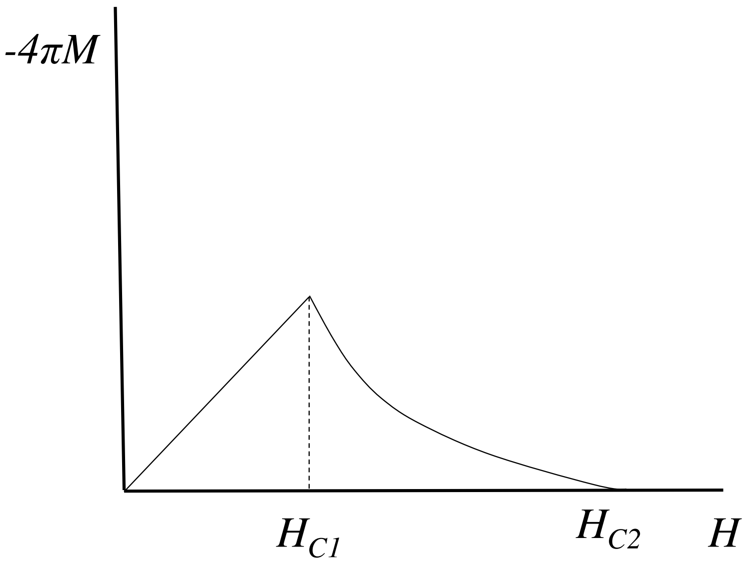
with $$u_n(\varphi + 2\pi) = u_n(\varphi)$$

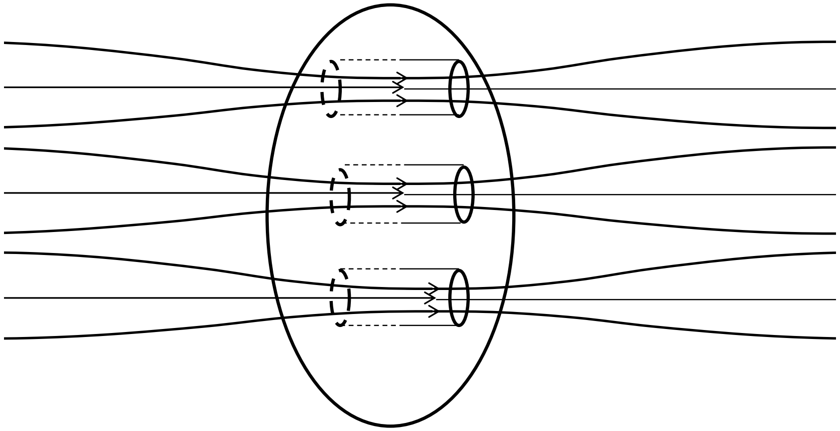
Band structure of the CPB

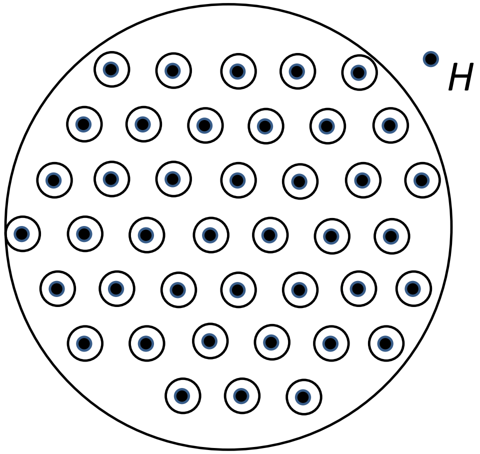


C. Vortices in superconductors









New charactristic length, *the coherence length* $\xi = \xi(T)$

$\xi_0 = \xi(0)$ is basically the radius of the Cooper pair

Estimate of this radius:

Energy cost to create an excitation in a metal is zero, but in a superconductor

$$E_F - \Delta < \frac{p^2}{2m} < E_F + \Delta$$

$\Delta << E_F$ ➡ $\delta p = 2\Delta / v_F$

Uncertainty principle ➡ $\delta x \propto \frac{\hbar}{\delta p} = \frac{\hbar v_F}{2\Delta}$ $\quad \xi_0 = \frac{\hbar v_F}{\pi \Delta}$

Temperature dependence is the same for $\xi(T)$ and $\lambda(T)$

What matters is λ_L/ξ_0 . From the Ginzburg-Landau theory

Pure metals $\lambda_L/\xi_0 < 1/\sqrt{2}$ ➡ Type I superconductors
Pippard theory

Alloys $\lambda_L/\xi_0 > 1/\sqrt{2}$ ➡ Type II superconductors
London theory

The supercurrent $\mathbf{J}_\mathbf{s}(\mathbf{r})$ is obtained from an average of $\mathbf{A}(\mathbf{r}')$
Over a region such that $|\mathbf{r}-\mathbf{r}'| < \xi_0$ in the Pippard theory

Condensation energy

Order parameter and penetration depth change abruptly at a surface

$$F_N - F_S = \frac{H_c^2}{8\pi}$$

If not

$$F_N - F_S = \frac{H_c^2}{8\pi} + \frac{H_c^2}{8\pi}\frac{(\lambda - \xi)S}{V}$$

In a cylinder the EM field penetrates the sample in tubes: *vortices*. Creation of as many vortices as possible to reduce the superconducting free energy. It is halted by vortex-vortex interaction.

Vortices

Energy per unit length of a vortex in a type II superconductor

$$\epsilon_l = \frac{1}{8\pi} \int_{r>\xi} dS(\lambda^2 \, |\nabla \times \mathbf{h}(\mathbf{r})|^2 + h^2(\mathbf{r}))$$

For $\kappa \equiv \lambda_L/\xi_0 >> 1\,,\quad \epsilon_l = \epsilon_0 \ln \kappa$ with $\epsilon_0 = \left(\frac{\phi_0}{4\pi\lambda}\right)^2$

For a distorted vortex tube with $\mathbf{u}(z) = [u_x(z), u_y(z)]$

$$\mathcal{F}_{\rm el} = \int dz \, \epsilon_l \left\{ \left[1 + \left(\frac{\partial \mathbf{u}}{\partial z}\right)^2 \right]^{1/2} - 1 \right\}$$

$$\approx \int dz \, \frac{\epsilon_l}{2} \left(\frac{\partial \mathbf{u}}{\partial z}\right)^2$$

Vortices

Fraction of electrons per unit length localized within the flux tube $2\pi\xi^2 N(E_F)\delta\epsilon$ where $\delta\epsilon \simeq \hbar v_F/\pi\xi$ and

$$N(E_F) = m_e k_F/2\pi^2\hbar^2$$

Change of the confined mass of electrons within the core

$$m_e \delta\epsilon/E_F$$

Linear density of mass of the vortex line is

$$m_l = \frac{2}{\pi^3} m_e k_F$$

This linear density can also be of other origins

Vortex-vortex interaction

Field produced at a given position by vortices 1 and 2

$$\mathbf{h}(\mathbf{r}) = \mathbf{h}_1(\mathbf{r}) + \mathbf{h}_2(\mathbf{r})$$

Vortex pair energy per unit length

$$E_l = \frac{\phi_0}{8\pi}\left[h_1(\mathbf{r}_1) + h_1(\mathbf{r}_2) + h_2(\mathbf{r}_1) + h_2(\mathbf{r}_2)\right]$$

Vortex pair interaction energy per unit length $\Delta E_l = \frac{\phi_0}{4\pi} h_1(\mathbf{r}_2)$

Vortex field $h(\mathbf{r}) = \frac{\phi_0}{2\pi\lambda^2} K_0\left(\frac{r}{\lambda}\right)$

$$\Delta E_l(r_{12}) = \frac{\phi_0^2}{8\pi^2\lambda^2} K_0\left(\frac{r_{12}}{\lambda}\right)$$

and $\mathbf{f}(\mathbf{r}_2) = -\nabla_2 \Delta E_l(r_{12}) = -\frac{\phi_0}{4\pi} \nabla_2 h_1(\mathbf{r}_2)$

Force on a vortex $\mathbf{f}(\mathbf{r}_2) = \mathbf{J}_1(\mathbf{r}_2) \times \frac{\mathbf{\Phi}_0}{c}$

In general, *Lorentz force* $\mathbf{f}_L(\mathbf{r}) = \mathbf{J}_s(\mathbf{r}) \times \frac{\mathbf{\Phi}_0}{c}$

Magnus force due to the motion relative to the superfluid velocity

$$\mathbf{f}_M(\mathbf{r}) = \rho_s[\mathbf{v}_s(\mathbf{r}) - \mathbf{v}_l] \times \frac{\mathbf{\Phi}_0}{c}$$

Dissipative and Hall effects on a stiff line $m_l \dot{\mathbf{v}}_l + \eta_l \mathbf{v}_l + \alpha_l \mathbf{v}_l \times \hat{\mathbf{z}} = \mathbf{f}_L$

$$\eta_l = \frac{\phi_0}{c} \rho_s \frac{\omega_0 \tau_r}{1 + \omega_0^2 \tau_r^2}$$

$$\alpha_l = \frac{\phi_0}{c} \rho_s \frac{\omega_0^2 \tau_r^2}{1 + \omega_0^2 \tau_r^2}$$

Elastic line

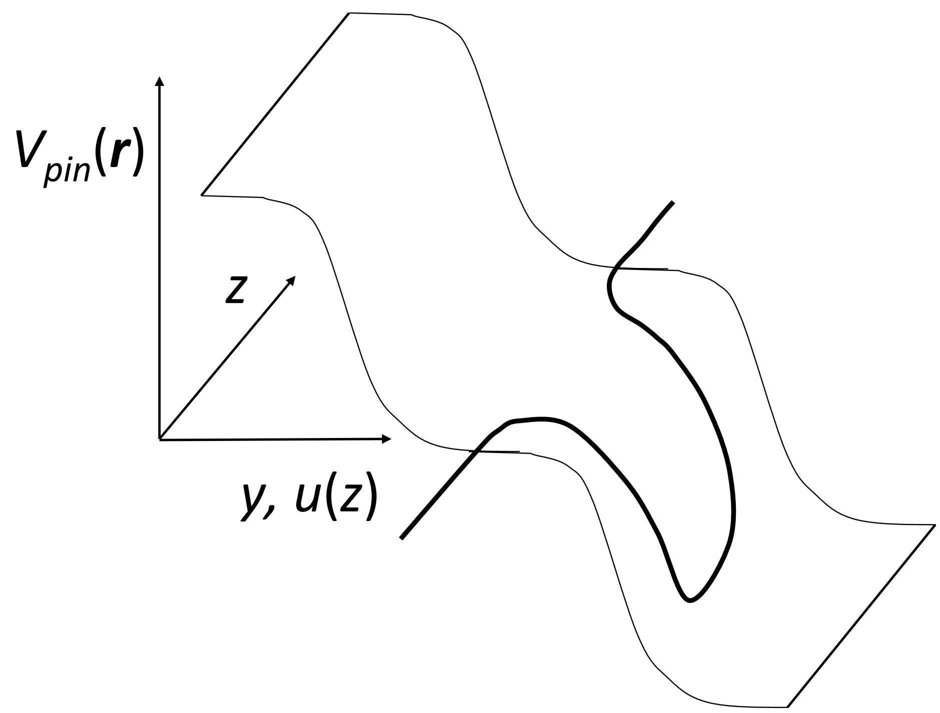
$$m_l \frac{\partial^2 \mathbf{u}(z,t)}{\partial t^2} + \eta_l \frac{\partial \mathbf{u}(z,t)}{\partial t} + \alpha_l \frac{\partial \mathbf{u}(z,t)}{\partial t} \times \hat{\mathbf{z}} - \epsilon_l \frac{\partial^2 \mathbf{u}(z,t)}{\partial z^2} + \frac{\partial V_{pin}(\mathbf{u}(z,t))}{\partial \mathbf{u}(z,t)} = \mathbf{f}_L$$

With the following replacements on the magnetic wall motion

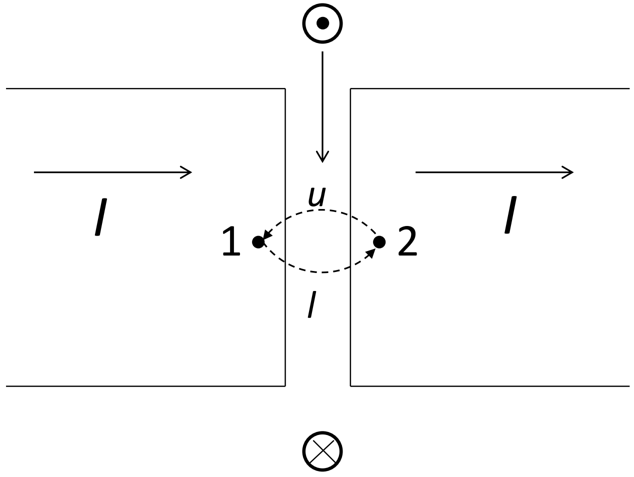
$$y \to z,\ u(y,t) \to \mathbf{u}(z,t),\ c_s \to \epsilon_l/m_l,$$
$$\tilde{V}_{pin} \to V_{pin}/\epsilon_l \text{ and } -(H/H_c)1/\zeta \to \mathbf{f}_L$$

We have the potential energy functional:

$$\mathcal{H}[\mathbf{u}(z,t)] = \int_{-\infty}^{+\infty} dz \left[\frac{\epsilon_l}{2} \left(\frac{\partial \mathbf{u}(z,t)}{\partial z} \right)^2 + V_{pin}(z, \mathbf{u}(z,t)) - \mathbf{f}_L \cdot \mathbf{u}(z,t) \right]$$



Phase slip



If one vortex crosses the junction:

$$\oint \nabla\theta \cdot d\mathbf{l} = \int_1^2 (\nabla\theta)_l \cdot d\mathbf{l} + \int_2^1 (\nabla\theta)_u \cdot d\mathbf{l} = 2\pi$$

$$\Downarrow$$

$$(\theta_1 - \theta_2)_u - (\theta_1 - \theta_2)_l \equiv \Delta\theta_u - \Delta\theta_l = 2\pi$$

If N vortices cross the junction:

$$\frac{d\Delta\theta}{dt} = 2\pi\frac{dN}{dt} \Rightarrow V = \phi_0\frac{dN}{dt} \Rightarrow V = \phi_0 n_v v_L d$$

D. Macroscopic Quantum Phenomena

Metastable configuration of the SQUID corresponds to a state of the condensate that carries zero current

$$|\Phi_i\rangle = |A_i\rangle\ |\psi_i\rangle$$

Stable configuration carries a finite current

$$|\Phi_f\rangle = |A_f\rangle\ |\psi_f\rangle$$

Total decaying state (caution)

$$|\Phi_D(t)\rangle \approx e^{-\frac{\gamma t}{2}}\ |A_i\rangle\ |\psi_i\rangle + \sqrt{(1-e^{-\gamma t})}\,|A_f\rangle\ |\psi_f\rangle$$

Another possibility is a bistable coherent oscillation between states carrying different currents

$$|\Phi_B(t)\rangle\ \approx\ a(t)\,|A_i\rangle\,|\psi_i\rangle + b(t)\,|A_f\rangle\,|\psi_f\rangle$$

They are both Schrödinger - cat like states

$$\Phi_B(\mathbf{r}, \mathbf{r}_1, ..., \mathbf{r}_N, t) = a(t)\, A_i(\mathbf{r})\, \psi_i(\mathbf{r}, \mathbf{r}_1, ..., \mathbf{r}_N) + b(t)\, A_f(\mathbf{r})\, \psi_f(\mathbf{r}, \mathbf{r}_1, ..., \mathbf{r}_N)$$

that differ from either a macroscopically occupied single particle state (the condensate wavefunction) or a Josephson Effect – like wavefunction

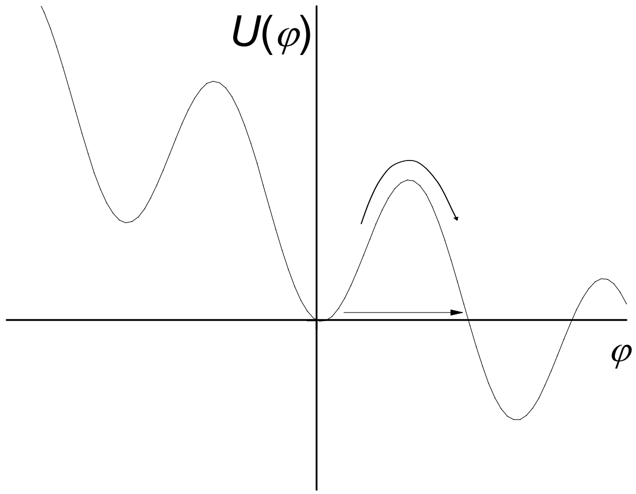
$$\varphi(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2, ..., \mathbf{x}_{N/2}, \mathbf{y}_{N/2}) = \prod_{i=1}^{N/2} \left[a_R^{(i)}\, \varphi_R(\mathbf{x}_i, \mathbf{y}_i) + a_L^{(i)}\, \varphi_L(\mathbf{x}_i, \mathbf{y}_i) \right]$$

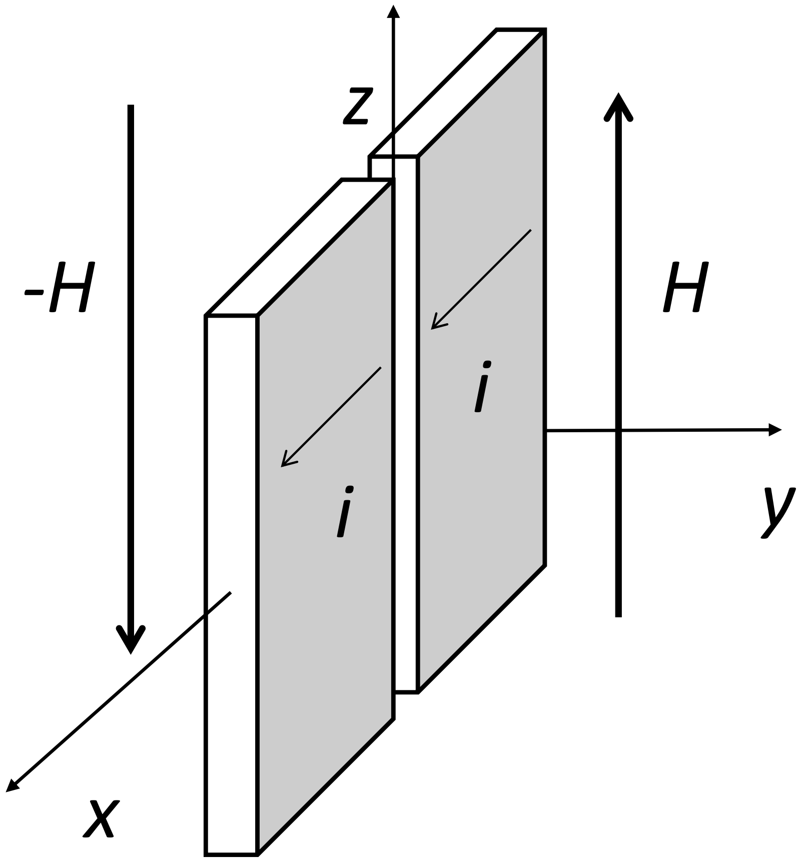
where $\varphi(\mathbf{x}_i, \mathbf{y}_i) = a_R^{(i)}\, \varphi_R(\mathbf{x}_i, \mathbf{y}_i) + a_L^{(i)}\, \varphi_L(\mathbf{x}_i, \mathbf{y}_i)$

Symbolically

$$\phi_C = a\, \phi_1^N + b\, \phi_2^N \qquad\qquad \phi_J = (a\, \phi_1 + b\, \phi_2)^{N/2}$$

Phase slip (phase representation)

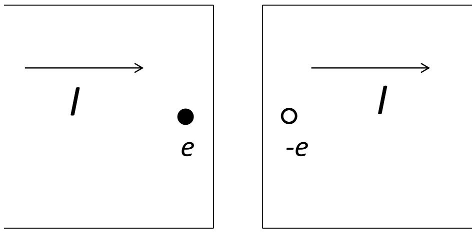




$$\phi_J = (a\,\phi_1 + b\,\phi_2)^{N/2}$$

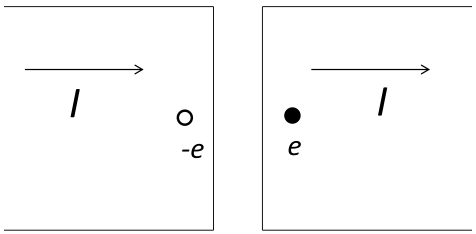
Phase slip (charge representation)

a)

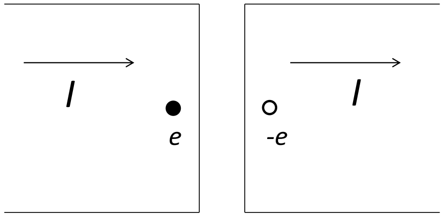


$$|0\rangle = |e, 0\rangle_L \otimes |-e, 0\rangle_R$$

b)



a)



$$|1\rangle = |e, -2e\rangle_L \otimes |-e, 2e\rangle_R$$

$$|\pm\rangle = |0\rangle \pm |1\rangle$$

b)