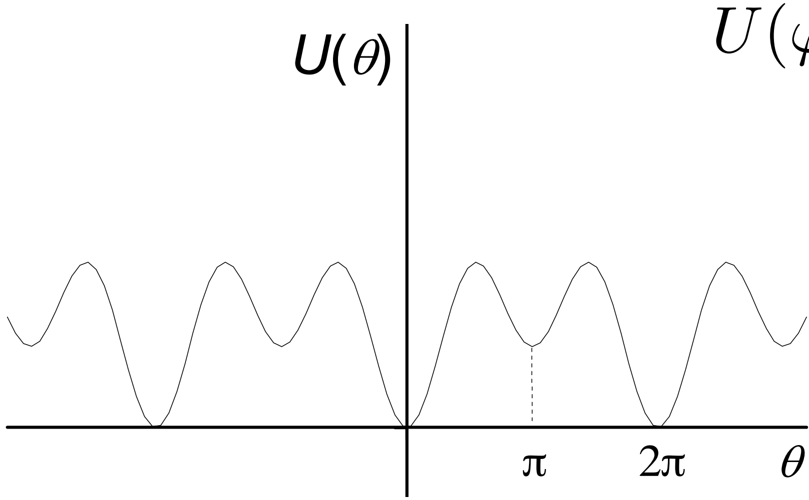
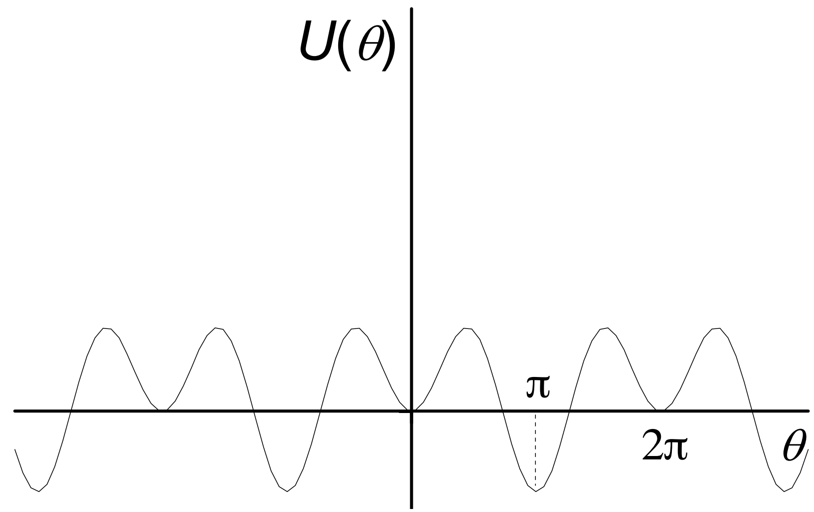
Quantum tunnelling (field theory)

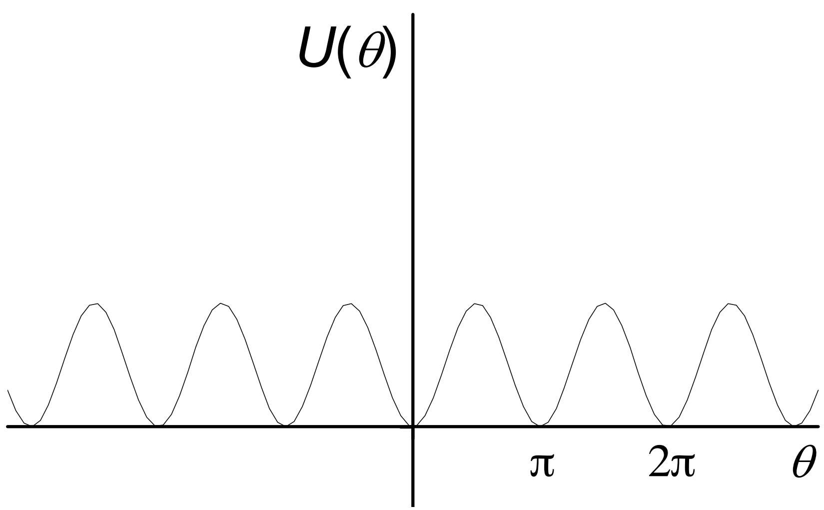
Metastable potential

$$U(\varphi) = g\sin^2\varphi + \lambda(1-\cos\varphi)$$



metastability $\lambda_c \equiv -2g < \lambda < 0$





Double sine-Gordon

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \nabla_D^2\varphi + g\sin 2\varphi + \lambda\sin\varphi = 0$$

$$U(\varphi) = g\sin^2\varphi + \lambda(1-\cos\varphi)$$

$$Z(\beta) = \mathrm{Tr}\exp -\beta H = \int \mathcal{D}\bar{\varphi}(\mathbf{r}') \int_{\bar{\varphi}(\mathbf{r}')}^{\bar{\varphi}(\mathbf{r}')} \mathcal{D}\varphi(\mathbf{r}',\tau')\exp - \frac{S_E[\varphi(\mathbf{r}',\tau')]}{\hbar}$$

$$S_E[\varphi(\mathbf{r}',\tau')] = \varphi_0^2 \int_{-\hbar\beta/2}^{+\hbar\beta/2} \int_{V_D} d\tau'\, d^D x \left\{ \frac{1}{2c^2}\left(\frac{\partial\varphi}{\partial\tau}\right)^2 + \frac{1}{2}|\nabla_D\varphi|^2 + U(\varphi) \right\}$$

where $\left[\frac{\varphi_0^2}{c}\right] L^{D-1} = [\hbar]$

zero temperature $\hbar\beta \to \infty$

Decay rate $$\frac{\Gamma_0}{V_D} = A_0 \exp - \frac{B_0}{\hbar}$$

Bounce is a spherically symmetric solution of the equation of motion

$$\frac{d^2\varphi}{d\rho^2} + \frac{D}{\rho}\frac{d\varphi}{d\rho} - g \sin 2\varphi - |\lambda| \sin\varphi = 0$$

"time variable" $\rho = \sqrt{c^2\tau^2 + r^2}$

$$\varphi_c(\rho = \infty) = 0 \qquad \text{and} \qquad \left.\frac{d\varphi_c}{d\rho}\right|_{\rho=0} = 0$$

Then we need to evaluate

$$B_0 = \frac{\varphi_0^2}{c} \int\limits_{-\infty}^{+\infty} c\, d\tau \int\limits_0^{\infty} r^{D-1}\, dr \int\limits_0^{2\pi} d\theta_1 \int\limits_0^{\pi} d\theta_2 ... \int\limits_0^{\pi} d\theta_{D-1} \sin^{D-2}\theta_{D-1} ... \sin\theta_2$$
$$\times \left\{ \frac{1}{2c^2} \left(\frac{\partial \varphi_c}{\partial \tau} \right)^2 + \frac{1}{2} |\nabla_D \varphi_c|^2 + U(\varphi_c) \right\}$$

$$B_0 = \frac{\varphi_0^2}{c} N(D) \int\limits_0^{\infty} d\rho\, \rho^D \left\{ \frac{1}{2} \left(\frac{\partial \varphi_c}{\partial \rho} \right)^2 + g \sin^2 \varphi_c - |\lambda| (1 - \cos\varphi_c) \right\}$$

For $D \geq 2$ we have $N(D) = 2\pi^{(D+1)/2} / \Gamma((D+1)/2)$

$$D = 1 \qquad \int\limits_0^{\infty} dr \to \int\limits_{-\infty}^{\infty} dx$$

Prefactor $$A_0 = \left(\frac{B_0}{2\pi\hbar}\right)^{(D+1)/2} c \sqrt{\frac{\det(-\nabla^2_{(D+1)} + U''(0))}{\det'(-\nabla^2_{(D+1)} + U''(\varphi_c))}}$$

Close to the instability $$|\lambda| \equiv 2g(1-\epsilon)$$

and the potential becomes $$U(\varphi) = -4g\sin^4\frac{\varphi}{2} + 4g\epsilon\sin^2\frac{\varphi}{2}$$

$$x \equiv \sqrt{2g}\rho \qquad \frac{d^2\varphi_c}{dx^2} + \frac{D}{x}\frac{d\varphi_c}{dx} = \sin\varphi_c(\cos\varphi_c - 1) + \epsilon\sin\varphi_c$$

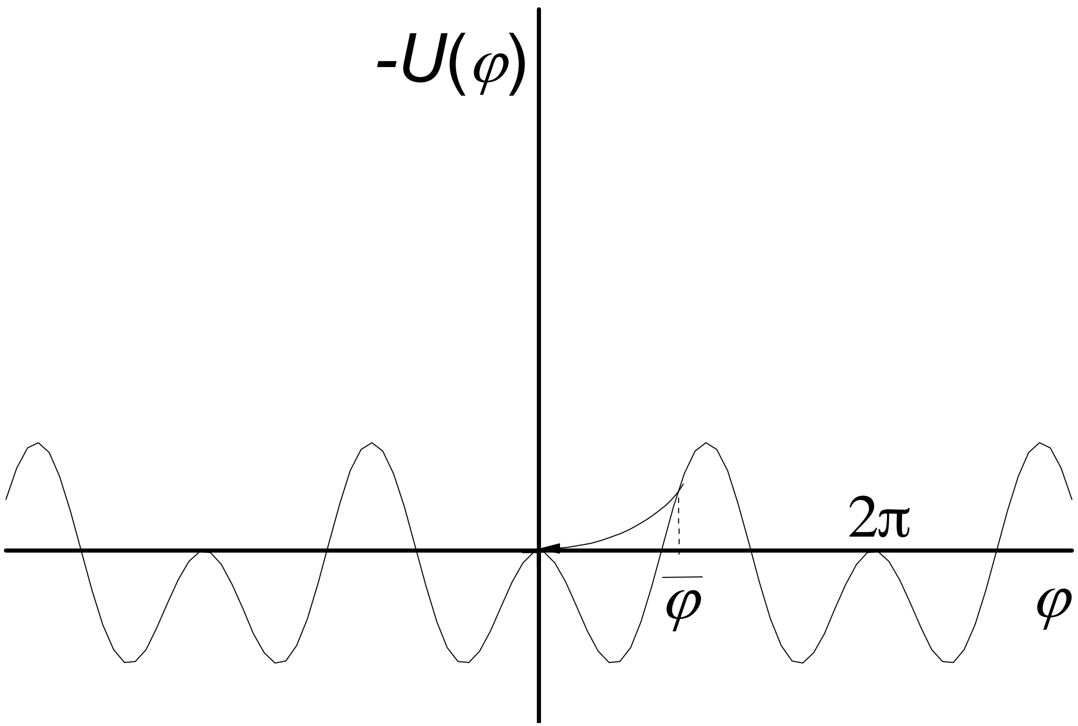
$$B = \frac{N(D)\varphi_0^2}{(2g)^{(D-1)/2}c}\int_0^\infty dx\, x^D \left\{\frac{1}{2}\left(\frac{\partial\varphi_c}{\partial x}\right)^2 - 2\sin^4\frac{\varphi_c}{2} + 2\epsilon\sin^2\frac{\varphi_c}{2}\right\}$$

$$\varphi_c(x=\infty) = 0 \;\text{ and }\; d\varphi_c/dx|_{x=0} = 0$$

Bounce analysis

$$\frac{d^2\varphi_c}{dx^2} + \frac{D}{x}\frac{d\varphi_c}{dx} = \sin\varphi_c(\cos\varphi_c - 1) + \epsilon\sin\varphi_c$$

$$\varphi_c(x=\infty) = 0 \;\text{ and }\; d\varphi_c/dx|_{x=0} = 0$$



Bounce action

$$B = P_D I_D(\epsilon) \qquad \text{where} \qquad P_D = \frac{2N(D)\varphi_0^2}{(2g)^{(D-1)/2}c} \qquad \text{and}$$

$$I_D(\epsilon) = \frac{1}{2}\int_0^\infty dx\, x^D \left\{2\varphi_c \sin^3\frac{\varphi_c}{2}\cos\frac{\varphi_c}{2} - 2\sin^4\frac{\varphi_c}{2} + \epsilon\left(2\sin^2\frac{\varphi_c}{2} - \frac{\varphi_c}{2}\sin\varphi_c\right)\right\}$$

For $D = 1$ or 2 dependence of the action on ϵ can be evaluated analytically

Approximated potential $\qquad U(\varphi) \approx 2g\epsilon\frac{\varphi^2}{2} - 2g\frac{\varphi^4}{8} - 2g\epsilon\frac{\varphi^4}{24}$

$$\tilde{x} \equiv \sqrt{\epsilon}x \;\text{ and }\; y(\tilde{x}) \equiv \varphi(\tilde{x})/\sqrt{2\epsilon}$$

$$B(\epsilon) \approx P_D \epsilon^{(3-D)/2} \tilde{I}_D \quad \text{with} \quad \tilde{I}_D = \int_0^\infty d\tilde{x}\, \tilde{x}^D \left\{ \frac{1}{2} \left(\frac{dy_c}{d\tilde{x}} \right)^2 + \frac{y_c^2}{2} - \frac{y_c^4}{4} \right\}$$

$$\frac{d^2 y_c}{d\tilde{x}^2} + \frac{D}{\tilde{x}} \frac{dy_c}{d\tilde{x}} = y_c - y_c^3 \qquad \tilde{I}_D = \int_0^\infty d\tilde{x}\, \tilde{x}^D \, \frac{y_c^4}{4}$$

Second variation problem

Eigenvalue equation $$-\frac{d^2\psi}{d\rho^2} - \frac{D}{\rho}\frac{d\psi}{d\rho} + \frac{k_{(D)}}{\rho^2}\psi + U''(\varphi_c)\psi = E_{(D)}\psi$$

with $k_{(1)} = m^2 \text{ with } m = 0, \pm 1, \pm 2, \ldots$

$k_{(2)} = \ell(\ell+1) \text{ with } \ell = 0, 1, \ldots$

$E_{(1)} = E_{nm} \text{ with } E_{(2)} = E_{n\ell}$

Rescaling $\tilde{x} \equiv \sqrt{\epsilon}x \text{ and } y(\tilde{x}) \equiv \varphi(\tilde{x})/\sqrt{2\epsilon}$

$$-\frac{d^2\psi}{d\tilde{x}^2} - \frac{D}{\tilde{x}}\frac{d\psi}{d\tilde{x}} + \frac{k_{(D)}}{\tilde{x}^2}\psi + U''(y_c)\psi = \frac{E_{(D)}}{2g\epsilon}$$

$$\sqrt{\frac{\det(-\nabla^2_{(D+1)} + 2g\epsilon)}{\det'(-\nabla^2_{(D+1)} + U''(\varphi_c))}} = (2g\epsilon)^{(D+1)/2} R_D$$

$$\frac{\Gamma}{V_D} \approx \left(\frac{g\epsilon B(\epsilon)}{\pi\hbar}\right)^{(D+1)/2} c\, R_D \exp -\frac{B(\epsilon)}{\hbar}$$

Thin-wall approximation (TWA)

Bounce solution close to the degeneracy $\varphi_c(\rho - R)$ $\rho = \sqrt{c^2\tau^2 + r^2}$

Real time behavior $\varphi_c(\sqrt{r^2 - c^2t^2} - R)$ $r_p^2(t) = R^2 + c^2t^2$

$$\rho \approx R \gg \zeta \equiv (2g)^{-1/2}$$

$$B = S_E(R_c) \qquad \text{where} \qquad \left.\frac{dS_E(R)}{dR}\right|_{R=R_c} = 0$$

$$S_E(R) = \frac{\varphi_0^2}{c} N(D) \int_0^\infty d\rho\, \rho^D \left\{ \frac{1}{2}\left(\frac{\partial \varphi_c}{\partial \rho}\right)^2 + g\sin^2\varphi_c - |\lambda|(1 - \cos\varphi_c) \right\}$$

$$\approx -\frac{2\varphi_0^2}{(D+1)c}\,|\lambda| N(D)\, R^{D+1} + \frac{\varphi_0^2}{2c}\, N(D)\, S_1 R^D$$

Where we have used

$$\varphi_c(\rho - R) = \begin{cases} \pi & \text{if} \quad \rho < R \\ \\ 0 & \text{if} \quad \rho > R \end{cases} \qquad \zeta = (2g)^{-1/2} \ \text{about} \ \rho \approx R$$

Then

$$\left.\frac{dS_E(R)}{dR}\right|_{R=R_c} = 0 \quad \Rightarrow \quad R_c = \frac{DS_1^{(D)}}{4|\lambda|} \quad \text{and} \quad B = \frac{\tilde{N}(D)\varphi_0^2}{c}\frac{S_1^{D+1}}{|\lambda|^D}$$

$$\tilde{N}(D) \equiv 2N(D)D^D/[(D+1)4^{D+1}] \qquad S_1 \equiv \int_0^\infty d\rho \left(\frac{\partial\varphi_c}{\partial\rho}\right)^2$$

Pre-factor analysis depends on $\quad (-\nabla^2_{(D+1)} + U''(\varphi_c))$

Finite temperatures: comments

$$S_{eff}[\varphi(\mathbf{r}', \tau')] = \varphi_0^2 \int_{-\hbar\beta/2}^{+\hbar\beta/2} d\tau' \int_{V_D} d^D x \left\{ \frac{1}{2c^2} \left(\frac{\partial \varphi}{\partial \tau} \right)^2 + \frac{|\nabla_D \varphi|^2}{2} + U(\varphi) \right\}$$

$$+ \frac{2\gamma\pi}{4\hbar^2\beta^2} \frac{\varphi_0^2}{c^2} \int_{V_D} d^D x \int_{-\hbar\beta/2}^{\hbar\beta/2} d\tau'' \int_{-\hbar\beta/2}^{\hbar\beta/2} d\tau' \frac{\{\varphi(x,\tau') - \varphi(x,\tau'')\}^2}{\sin^2\left(\pi(\tau' - \tau'')/\hbar\beta\right)}$$

Bonce solution

$$\left. \frac{\delta S_{eff}}{\delta \varphi} \right|_{\varphi_c} = \frac{1}{c^2} \frac{\partial^2 \varphi_c}{\partial \tau^2} + \nabla_D^2 \varphi_c - \frac{\partial U}{\partial \varphi_c}$$

$$- \frac{2\gamma\pi}{\hbar^2\beta^2 c^2} \int_{-\hbar\beta/2}^{\hbar\beta/2} d\tau'' \frac{[\varphi_c(x,\tau') - \varphi_c(x,\tau'')]}{\sin^2\left(\pi(\tau' - \tau'')/\hbar\beta\right)} = 0$$

Second variation

$$\hat{D}_\beta \varphi(x,\tau') = -\frac{1}{c^2}\frac{\partial^2 \varphi(x,\tau')}{\partial \tau'^2} - \nabla_D^2 \varphi(x,\tau') + U''(\varphi_c)\varphi(x,\tau') + \hat{O}_\beta \varphi(x,\tau') = kq(\tau')$$

$$\hat{O}_\beta \varphi(x,\tau') = \frac{2\gamma\pi}{\hbar^2\beta^2 c^2} \int_{-\hbar\beta/2}^{\hbar\beta/2} d\tau'' \, \frac{[\varphi(x,\tau') - \varphi(x,\tau'')]}{\sin^2\left(\pi(\tau' - \tau'')/\hbar\beta\right)}$$

$$S_{eff}(\varphi_n(x)) = \hbar\omega_0 \beta B_0 \int_{V_D} d^D x \sum_{n=-\infty}^{\infty} \Big[(\nu_n^2 + 2\alpha|\nu_n|)\varphi_n(x)\varphi_{-n}(x) + \nabla_D \varphi_n(x) \cdot \nabla_D \varphi_{-n}(x) - [U(\varphi(x,\tau))]_n \Big]$$