Damped harmonic oscillator

Reduced density operator as a function of time

$$\tilde{\rho}(x,y,t) = \int\int dx'dy'\,\mathcal{J}(x,y,t;x',y',0)\,\tilde{\rho}(x',y',0)$$

Super-propagator

$$\mathcal{J}(x,y,t;x',y',0) = \int\int\int d\mathbf{R}'d\mathbf{Q}'d\mathbf{R}\,\Big\{K(x,\mathbf{R},t;x',\mathbf{R}',0)$$
$$\times K^*(y,\mathbf{R},t;y',\mathbf{Q}',0)\,\tilde{\rho}_R(\mathbf{R}',\mathbf{Q}',0)\Big\}.$$

And if $\quad S[x(t'),\mathbf{R}(t')] = \tilde{S}_0[x(t')] + S_I[x(t'),\mathbf{R}(t')] + S_R[\mathbf{R}(t')]$

$$\mathcal{J}(x,y,t;x',y',0) = \int_{x'}^{x}\int_{y'}^{y}\mathcal{D}x(t')\,\mathcal{D}y(t')\;\exp\left\{\frac{i}{\hbar}\tilde{S}_0[x(t')]\right\}\;\exp\left\{-\frac{i}{\hbar}\tilde{S}_0[y(t')]\right\}$$
$$\times\mathcal{F}[x(t'),y(t')]$$

In general
$$\mathcal{J}(x,x',t;y,y',0)=\int_{x'}^{x}\int_{y'}^{y}\mathcal{D}x(t')\mathcal{D}y(t')\exp\frac{i}{\hbar}\left\{S_0[x(t')]-S_0[y(t')]-\right.$$
$$\left.-\int_0^t\int_0^\tau d\tau d\sigma[x(\tau)-y(\sigma)]M\gamma(\tau-\sigma)[\dot{x}(\tau)+\dot{y}(\sigma)]\right\}\times$$
$$\times\exp-\frac{1}{\hbar}\int_0^t\int_0^\tau d\tau d\sigma[x(\tau)-y(\tau)]D(\tau-\sigma)[x(\sigma)-y(\sigma)]$$

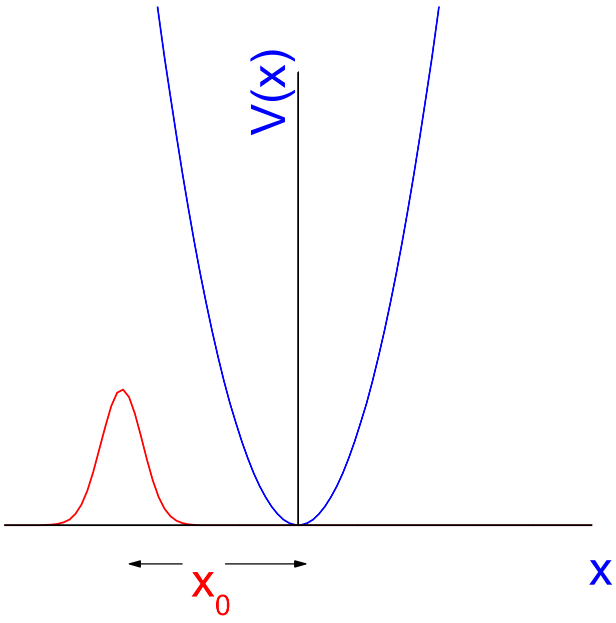
With boundary condition $\mathcal{J}(x,y,0^+;x',y',0)=\delta(x-x')\delta(y-y')$

Under the previous choice of the spectral function the final super-propagator is

$$\boxed{\begin{aligned}&J(x,x',t;y,y',0)=\int_{x'}^{x}\int_{y'}^{y}\mathcal{D}x(t')\mathcal{D}y(t')\times\\&\times\exp\frac{i}{\hbar}\left\{S_0[x(t')]-S_0[y(t')]-M\gamma\int_0^t(x\dot{x}-y\dot{y}+x\dot{y}-y\dot{x})dt'\right\}\times\\&\times\exp-\frac{2M\gamma}{\pi\hbar}\int_0^\Omega d\omega\,\omega\,\coth\frac{\hbar\omega}{2k_BT}\int_0^t\int_0^\tau d\tau d\sigma[x(\tau)-y(\tau)]\cos\omega(\tau-\sigma)[x(\sigma)-y(\sigma)]\end{aligned}}$$

Applications

Damped harmonic oscillator



$$\psi(x') = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp \frac{ipx'}{h} \exp - \frac{x'^2}{4\sigma^2}$$

Initial state

Super-propagator

$$\mathcal{J}(x, x', t; y, y', 0) = \int_{x'}^{x} \int_{y'}^{y} \mathcal{D}x(t')\mathcal{D}y(t') \, \exp\frac{i}{\hbar}\tilde{S}[x(\tau), y(\tau)]\times$$

$$\times\exp - \frac{1}{\hbar}\phi[x(\tau), y(\tau)]$$

where $\tilde{S}[x(\tau), y(\tau)] = \int_{0}^{t} \tilde{L}(x, \dot{x}, y, \dot{y}) \, d\tau - M\gamma \int_{0}^{t} (x\dot{x} - y\dot{y})dt'$

$$\tilde{L}(x, \dot{x}, y, \dot{y}) = \frac{1}{2}M\dot{x}^2 - \frac{1}{2}M\dot{y}^2 - \frac{1}{2}M\omega_0^2 x^2 + \frac{1}{2}M\omega_0^2 y^2 - M\gamma x\dot{y} + M\gamma y\dot{x}$$

and $\phi[x(\tau), y(\tau)] = \exp-\frac{2M\gamma}{\pi\hbar} \int_{0}^{\Omega} d\nu \, \nu \, \coth\frac{\hbar\nu}{2k_B T} \times$

$$\times \int_{0}^{t} \int_{0}^{\tau} d\tau d\sigma [x(\tau) - y(\tau)] \cos\nu(\tau - \sigma) [x(\sigma) - y(\sigma)] \, d\tau \, d\sigma$$

Stationary paths

$$\left.\frac{\delta \tilde{S}}{\delta x}\right|_{x=x_c, y=y_c} = \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{x}} - \frac{\partial \tilde{L}}{\partial x} = M\ddot{x}_c + 2M\gamma\dot{y}_c + M\omega_0^2 x_c = 0,$$

$$\left.\frac{\delta \tilde{S}}{\delta y}\right|_{x=x_c, y=y_c} = \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{y}} - \frac{\partial \tilde{L}}{\partial y} = M\ddot{y}_c + 2M\gamma\dot{x}_c + M\omega_0^2 y_c = 0$$

New variables $\quad q(\tau) \equiv \dfrac{x(\tau)+y(\tau)}{2} \quad$ and $\quad \xi(\tau) \equiv x(\tau) - y(\tau)$

Then

$$\ddot{q}_c + 2\gamma\dot{q}_c + \omega_0^2 q_c = 0$$

$$\ddot{\xi}_c - 2\gamma\dot{\xi}_c + \omega_0^2 \xi_c = 0$$

Solutions

$$q_c(\tau) = (\sin\omega t)^{-1}\left\{q e^{\gamma t}\sin\omega\tau + q'\sin\omega(t-\tau)\right\}e^{-\gamma\tau},$$

$$\xi_c(\tau) = (\sin\omega t)^{-1}\left\{\xi e^{-\gamma t}\sin\omega\tau + \xi'\sin\omega(t-\tau)\right\}e^{\gamma\tau}$$

$$\omega \equiv \sqrt{\omega_0^2 - \gamma^2} \quad \text{if} \quad \omega_0 > \gamma \qquad \text{and} \qquad \omega \equiv i\sqrt{\gamma^2 - \omega_0^2} \quad \text{if} \quad \omega_0 < \gamma$$

New variations $\tilde{q}(\tau) \equiv q(\tau) - q_c(\tau)$ $\tilde{\xi}(\tau) \equiv \xi(\tau) - \xi_c(\tau)$

Super-propagator

$$\mathcal{J}(q,\xi,t;q',\xi',0) = \exp\left\{\frac{i}{\hbar}\tilde{S}_c\right\} \exp -\frac{1}{\hbar}\left\{A(t)\xi^2 + B(t)\xi\xi' + C(t)\xi'^2\right\}\times$$
$$\times G(q,\xi,t;q',\xi',0)$$

with $\tilde{S}_c = K(t)[q\xi + q'\xi'] - L(t)q'\xi - N(t)q\xi' - M\gamma[q\xi - q'\xi']$

where

$$K(t) = M\omega\cot\omega t, \quad L(t) = \frac{M\omega e^{-\gamma t}}{\sin\omega t} \quad \text{and} \quad N(t) = \frac{M\omega e^{\gamma t}}{\sin\omega t}$$

and $A(t), B(t)$ and $C(t)$ obey $f(t) = \dfrac{M\gamma}{\pi}\displaystyle\int_0^\Omega d\nu\,\nu\,\coth\frac{\hbar\nu}{2k_BT}\,f_\nu(t)$

with $$A_\nu(t) = \frac{e^{-2\gamma t}}{\sin^2 \omega t} \int\limits_0^t \int\limits_0^t \sin\omega\tau \cos\nu(\tau-\sigma) \sin\omega\sigma \, e^{\gamma(\tau+\sigma)} \, d\tau \, d\sigma$$

$$B_\nu(t) = \frac{2e^{-2\gamma t}}{\sin^2 \omega t} \int\limits_0^t \int\limits_0^t \sin\omega\tau \cos\nu(\tau-\sigma) \sin\omega(t-\sigma) \, e^{\gamma(\tau+\sigma)} \, d\tau \, d\sigma$$

$$C_\nu(t) = \frac{1}{\sin^2 \omega t} \int\limits_0^t \int\limits_0^t \sin\omega(t-\tau) \cos\nu(\tau-\sigma) \sin\omega(t-\sigma) \; e^{\gamma(\tau+\sigma)} \, d\tau \, d\sigma$$

Then $$G(q,\xi,t;q',\xi',0) = \int\limits_0^0 \int\limits_0^0 \mathcal{D}\tilde{q}(\tau)\mathcal{D}\tilde{\xi}(\tau) \; \exp\frac{i}{\hbar}\tilde{S}[\tilde{q}(\tau),\tilde{\xi}(\tau)]\times$$
$$\times\exp -\frac{1}{\hbar}\phi_T[\tilde{\xi}(\tau),\tilde{\xi}(\tau)] \; \exp\frac{2}{\hbar}\phi_T[\xi_c(\tau),\tilde{\xi}(\tau)]$$

where

$$\phi_T[f(\tau),g(\tau)] = \frac{M\gamma}{\pi} \int\limits_0^\Omega d\nu \, \nu \; \coth\frac{\hbar\nu}{2k_B T} \int\limits_0^t \int\limits_0^t d\tau \, d\sigma \; f(\tau)\cos\nu(\tau-\sigma) g(\sigma) \, d\tau \, d\sigma$$

Prefactor of the form $$N\int \delta U \exp -\frac{1}{2}U^T\mathcal{M}U \exp -\mathcal{A}U$$

where $$U^T = (\tilde{q}_1, ..., \tilde{q}_N, \tilde{\xi}_1, ..., \tilde{\xi}_N)$$

and $$\delta U = d\tilde{q}_1...d\tilde{q}_N d\tilde{\xi}_1...d\tilde{\xi}_N$$

We have defined $$\mathcal{A} = \begin{pmatrix} 0 \\ a \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} 0 & p \\ p & r \end{pmatrix}$$

Using that $$N\int \delta U \exp -\frac{1}{2}U^T\mathcal{M}U \exp -\mathcal{A}U = \frac{1}{\sqrt{\det \mathcal{M}}} \exp \frac{1}{2}\mathcal{A}\mathcal{M}^{-1}\mathcal{A}$$

And $\mathcal{A}\mathcal{M}^{-1}\mathcal{A} = 0$, we have

$$\mathcal{J}(q, \xi, t; q', \xi', 0) = G(t) \exp \frac{i}{\hbar}\Big\{[K(t) - M\gamma]q\xi + [K(t) + M\gamma]q'\xi' -$$
$$-L(t)q'\xi - N(t)q\xi'\Big\} \exp -\frac{1}{\hbar}\Big\{A(t)\xi^2 + B(t)\xi\xi' + C(t){\xi'}^2\Big\}$$

Now, using $$\tilde{\rho}(q,\xi,t) = \int\int dq'\,d\xi'\; \mathcal{J}(q,\xi,t;q',\xi',0)\,\tilde{\rho}(q',\xi',0)$$

and $$\mathcal{J}(q,\xi,t;q',\xi',0) = G(t)\exp\frac{i}{\hbar}\Big\{[K(t)-M\gamma]q\xi + [K(t)+M\gamma]q'\xi' - $$
$$-L(t)q'\xi - N(t)q\xi'\Big\}\exp -\frac{1}{\hbar}\Big\{A(t)\xi^2 + B(t)\xi\xi' + C(t){\xi'}^2\Big\}$$

We have for the initial condition

$$\tilde{\rho}(q',\xi',0) = \frac{1}{(2\pi\sigma^2)^{1/2}}\;\exp\frac{ip\,\xi'}{\hbar}\;\exp -\frac{q'^2}{2\sigma^2}\;\exp -\frac{\xi'^2}{8\sigma^2}$$

the result $$\tilde{\rho}(q,\xi,t) = G(t)\sqrt{\frac{\pi\hbar^2}{N^2(t)\sigma^2(t)}}\exp -\frac{1}{2\sigma^2(t)}(q-x_0(t))^2\times$$
$$\exp -F(t)\xi^2\;\;\exp\frac{i}{\hbar}D(q,p,t)\xi$$

$$\tilde{\rho}(q,\xi,t) = G(t)\sqrt{\frac{\pi\hbar^2}{N^2(t)\sigma^2(t)}}\,\exp -\frac{1}{2\sigma^2(t)}(q - x_0(t))^2 \times$$
$$\exp -F(t)\xi^2 \;\; \exp \frac{i}{\hbar} D(q,p,t)\xi$$

With the following time dependent functions

$$G(t) = \frac{N(t)}{\sqrt{2}\pi\hbar}, \quad \sigma^2(t) = \frac{\sigma^2 K_1^2(t) + 2\hbar C_1^2(t)}{N^2(t)}, \qquad x_0(t) = \frac{p}{N(t)}$$

$$F(t) = \frac{A(t)}{\hbar} + \frac{\sigma^2 L^2(t)}{2\hbar^2} - \frac{(\sigma^2 K_1(t)L(t) - \hbar B(t))^2}{2\hbar^2\sigma^2(t)N^2(t)}$$

$$D(q,p,t) = K_2(t)q - \frac{(\sigma^2 K_1(t)L(t) - \hbar B(t))}{\sigma^2(t)N(t)}(q - x_0(t))$$

$$C_1(t) \equiv C(t) + \frac{\hbar}{8\sigma^2}\,, \quad K_1(t) \equiv K(t) + M\gamma \quad \text{and} \quad K_2(t) \equiv K(t) - M\gamma$$

General results

Time evolution

$$\tilde{\rho}(x,x,t) = \left(\frac{1}{2\pi\sigma^2(t)}\right)^{1/2} \exp - \frac{1}{2\sigma^2(t)}\,(x - x_0(t))^2$$

Center of the wavepacket

$$\boxed{x_0(t) = \frac{p}{M\omega}\sin\omega t\; e^{-\gamma t}}$$

Width at equilibrium

$$\boxed{\sigma^2(\infty) = \frac{\hbar}{\pi}\int_0^\infty d\nu \coth\frac{\hbar\nu}{2kT}\left(\frac{1}{M}\frac{2\gamma\nu}{\left(\omega_0^2 - \nu^2\right)^2 + 4\gamma^2\nu^2}\right)}$$

Fluctuation-dissipation theorem

$$\boxed{\sigma^2(\infty) = \frac{\hbar}{\pi}\int_0^\infty d\nu \coth\frac{\hbar\nu}{2kT}\chi''(\nu)}$$

Behavior of the width at zero temperature for any value of the damping constant

$$\sigma^2(\infty) = \frac{\hbar}{2M\omega_0} f(\alpha) \qquad \left(\alpha \equiv \frac{\gamma}{\omega_0}\right)$$

$$f(\alpha) = \begin{cases} \frac{1}{\sqrt{1-\alpha^2}} \left(1 - \frac{2}{\pi} \tan^{-1} \frac{\alpha}{\sqrt{1-\alpha^2}}\right) & \text{se} \quad \alpha < 1 \\ \\ \frac{1}{\sqrt{\alpha^2-1}} \frac{1}{\pi} \ln \left| \frac{\alpha+\sqrt{\alpha^2-1}}{\alpha-\sqrt{\alpha^2-1}} \right| & \text{se} \quad \alpha > 1 \end{cases}$$

Finite damping always reduces the width

If $\omega_0 \gg \gamma$ $\quad A_\nu(t), B_\nu(t),$ and $C_\nu(t) \propto \delta(\nu - \omega_0)$

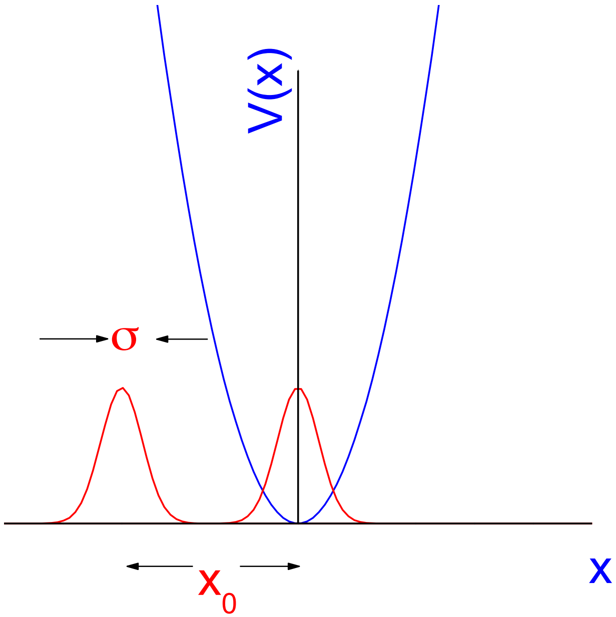
Proportionality constants

$$g^{(\alpha)}(t) \left\{ \begin{aligned} g^{(A)}(t) &= \frac{\pi}{\sin^2 \omega_0 t} \int_0^t d\tau \sin^2 \omega_0 \tau, \\ g^{(B)}(t) &= \frac{2\pi}{\sin^2 \omega_0 t} \int_0^t d\tau \sin \omega_0 \tau \sin \omega_0 (t - \tau), \\ g^{(C)}(t) &= \frac{\pi}{\sin^2 \omega_0 t} \int_0^t d\tau \sin^2 \omega_0 (t - \tau) \end{aligned} \right.$$

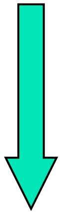
Then $\left\{ \begin{matrix} A(t) \\ B(t) \\ C(t) \end{matrix} \right\} = M\gamma\, \omega_0 \coth \dfrac{\hbar \omega_0}{2k_B T}\, g^{(\alpha)}(t)$

And $\Phi[x(t'), y(t')] = \dfrac{1}{\hbar} M\gamma\, \omega_0 \coth \dfrac{\hbar \omega_0}{2k_B T} \displaystyle\int_0^t dt' [x(t') - y(t')]^2$

Decoherence



Initial state



$$\psi(x) = \psi_1(x) + \psi_2(x) = \tilde{\mathcal{N}} \left[\exp - \frac{x^2}{4\sigma^2} + \exp - \frac{(x + x_0)^2}{4\sigma^2} \right]$$

Study of the interference between two wave packets

Initial density operator

$$\rho(x',y',0) = \rho_1(x',y',0) + \rho_2(x',y',0) + \rho_{int}(x',y',0)$$

Time evolution $\qquad \tilde{\rho}(x,t) = \tilde{\rho}_1(x,t) + \tilde{\rho}_2(x,t) + \tilde{\rho}_{int}(x,t)$

Linearity of the time evolution

$$\tilde{\rho}_{int}(x,x,t) = \int dx'dy' J(x,x,t;x',y',0)\tilde{\rho}_{int}(x',y',0)$$

Without damping $\qquad \rho_{int}(x,t) = 2\sqrt{\rho_1(x,t)\rho_2(x,t)}\cos\phi(x,t)$

Where $\qquad \rho_{int}(x,t=n\pi/\omega_0+\pi/2\omega_0) = \cos\left(\frac{q_0}{\sigma^2}x\right)\exp -\frac{x^2}{\sigma^2}$

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi(x,t)$$

Damped case $\tilde{\rho}_{int}(x,t) = 2\sqrt{\tilde{\rho}_1(x,t)}\sqrt{\tilde{\rho}_2(x,t)}\cos\phi(x,t)\exp - f(t)$

Attenuation factor $\boxed{\exp - f(t) \approx \exp - \Gamma t}$ where $f(t) = \dfrac{q_0^2 \alpha I_R(\theta)}{8\sigma^2 Q(\theta)}$

The time dependent functions are

$$q(\theta) = q_0 \left[\frac{\alpha \sin S\theta}{S} + \cos S\theta \right] \exp - \alpha\theta,$$

$$C_R(\theta,\lambda) \equiv \frac{1}{\sin^2 S\theta} \int_0^\theta \int_0^\theta \sin\left[S(\theta - \theta_1)\right] \cos\left[\lambda(\theta_1 - \theta_2)\right] \times$$
$$\times \sin\left[S(\theta - \theta_2)\right] \exp\left[\alpha(\theta_1 + \theta_2)\right] d\theta_1\, d\theta_1,$$

$$I_R(\theta) = \frac{4}{\pi} \int_0^{\lambda_c} d\lambda\, \lambda\, C_R(\theta,\lambda) \coth(\kappa\lambda) \ ,$$

$$Q(\theta) \equiv 1 + \alpha I_R(\theta) + (\alpha + S \cot S\theta)^2 \qquad \text{and}$$

$$\sigma^2(\theta) = \frac{\sigma^2 Q(\theta) \sin^2(S\theta) \exp - (2\alpha\theta)}{S^2}$$

Where we have defined $\alpha \equiv \dfrac{\gamma}{\omega_0}$, $S \equiv \dfrac{\omega}{\omega_0}$, $\theta \equiv \omega_0 t$, $\kappa \equiv \dfrac{\hbar\omega_0}{2k_B T}$, $\lambda_c \equiv \dfrac{\Omega}{\omega_0}$

Attenuation factor $\boxed{\exp -f(t) \approx \exp -\Gamma t}$

Decoherence rate

$$\boxed{\Gamma = \begin{cases} \text{high temperatures} \quad (\kappa \ll 1) \begin{cases} \frac{2NkT}{\hbar\omega_0}\gamma & \text{if} \;\; \gamma \ll \omega_0 \\ \frac{2NkT}{\hbar\omega_0}\frac{\omega^2}{2\gamma} & \text{if} \;\; \gamma \gg \omega_0 \end{cases} \\ \\ \text{low temperatures} \quad (\kappa \gg 1) \begin{cases} N\gamma & \text{if} \;\; \gamma \ll \omega_0 \\ N\frac{\omega_0^2}{2\gamma} & \text{if} \;\; \gamma \gg \omega_0 \end{cases} \end{cases}}$$

$N \equiv x_0^2/4\sigma^2$ is also the average number of energy quanta initially in the system

$\kappa \equiv \hbar\omega_0/kT$ is an inverse dimensionless temperature

Interpretation

Initial state of the universe when the environment is at zero temperature

$$|\phi_1\rangle \approx \{|\psi_0\rangle + |\psi_z\rangle\} \otimes |0\rangle$$

Final equilibrium state state of the universe

$$|\phi_f\rangle = |\psi_o\rangle \otimes |N\rangle$$

State of the universe after one quantum of energy is delivered to the bath

$$|\phi_1\rangle \approx |\tilde{\psi}_z\rangle \otimes |1\rangle + |\tilde{\psi}_0\rangle \otimes |0\rangle$$

Partial trace over the states of the environment

$$\tilde{\rho} \equiv \mathrm{tr}_R |\phi_1\rangle\langle\phi_1| = |\tilde{\psi}_z\rangle\langle\tilde{\psi}_z| + |\tilde{\psi}_0\rangle\langle\tilde{\psi}_0|$$

Interpretation for finite temperatures

Time evolution of the occupation numbers at finite temperature

$$\dot{n}_1 = -An_1 - An(\omega_0)(n_1 - n_2)$$

$$\dot{n}_2 = An_1 + An(\omega_0)(n_1 - n_2)$$

$$E_1 - E_2 \approx \hbar\omega_0 \qquad A \propto (\tau/N)^{-1} \qquad n(\omega_0) = \frac{1}{\exp \frac{\hbar\omega_0}{k_B T} - 1}$$

Solution $\quad n_1 - n_2 \;\propto\; \exp -[2n(\omega_0) + 1]At$

Decoherence time $\quad \tau_d \;\propto\; \dfrac{1}{2n(\omega_0) + 1}\dfrac{\tau}{N}$

which at high temperatures is $\quad \tau_d \approx \dfrac{\hbar\omega_0}{2Nk_B T}\tau$

Pointer basis

Pointer states $|P_i\rangle$

Environment states $\{|E_i^{(n)}\rangle\}$

Observable state to be measured $|\psi\rangle = \sum_i a_i |\varphi_i\rangle \qquad \hat{\mathcal{O}}|\varphi_i\rangle = \mathcal{O}_i|\varphi_i\rangle$

Initial apparatus state $|A_0\rangle = |P_0\rangle \otimes |E_0\rangle$

where $|E_0\rangle = |E_0^{(1)}\rangle \otimes |E_0^{(2)}\rangle \otimes \ldots$

Then $|\Psi\rangle = |\psi\rangle \otimes |P_0\rangle \otimes |E_0\rangle = \sum_i a_i(|\varphi_i\rangle \otimes |P_0\rangle \otimes |E_0\rangle)$

evolves to $|\Psi(t)\rangle = \sum_i a_i(|\varphi_i\rangle \otimes |P_i\rangle \otimes |E_i\rangle)$

and $\rho_{SP} \equiv \mathrm{tr}_R \rho = \sum_i |a_i|^2 |\varphi_i\rangle\langle\varphi_i| \otimes |P_i\rangle\langle P_i|$

Internal decoherence

Estimate from the master equation $\tau_D \approx \gamma^{-1} \left(\frac{\lambda_T}{\Delta x}\right)^2$ where

$\lambda_T = \hbar/\sqrt{2Mk_BT}$ is the de Broglie thermal wavelength

But the exact solution gives $\tau_D = \frac{1}{\gamma}\left(\frac{\sigma}{\xi}\right)^2 \left(\exp\frac{\hbar\omega_0}{k_BT} - 1\right)$

$$= \begin{cases} \frac{1}{\gamma}\frac{\hbar\omega_0}{k_BT}\left(\frac{\sigma}{\xi}\right)^2 & \text{if} \quad k_BT \gg \hbar\omega_0 \\ \frac{1}{\gamma}\left(\frac{\sigma}{\xi}\right)^2 \exp\frac{\hbar\omega_0}{k_BT} & \text{if} \quad k_BT \ll \hbar\omega_0. \end{cases}$$

which depends on the initial preparation off-diagonal distance ξ

For the purity $\mathrm{Tr}\tilde{\rho}^2(x,y,t) \approx 1 - 4\left(\exp\frac{\hbar\omega_0}{k_BT} - 1\right)^{-1}\gamma t$

and then $\tau_c = \left(\exp\frac{\hbar\omega_0}{k_BT} - 1\right)/4\gamma$