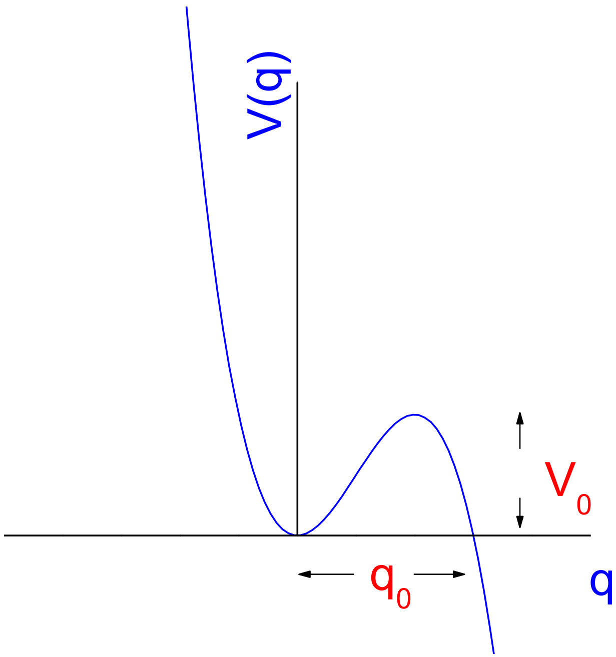
Dissipative quantum tunnelling (T=0)

Metastable potential



$$V(q) = \frac{1}{2} M \omega_0^2 q^2 - \lambda q^3 \qquad (\lambda > 0)$$

$$\psi(q,t) \propto e^{-i(E_R + iE_I)t/\hbar}$$

$$\psi^*(q,t)\psi(q,t) \propto e^{-2|E_I|t/\hbar} \quad \Rightarrow \quad \Gamma = \frac{2|E_I|}{\hbar}$$

Undamped point particles: standard example

$$V(q) = \frac{1}{2} M\omega_0^2 q^2 - \frac{1}{6} M\lambda\, q^3 \qquad\qquad (\lambda > 0)$$

$$\Gamma_0 = \frac{2\,|\mathrm{Im} E_0^{(0)}|}{\hbar} = \left(\frac{B_0}{2\pi\hbar M}\right)^{1/2} \left|\frac{\det(-M\partial_t^2 + M\omega_0^2)}{\det'(-M\partial_t^2 + V''(q_c^{(0)}))}\right|^{1/2} \exp - \frac{B_0}{\hbar}$$

$$\left.\frac{\delta S_E}{\delta q}\right|_{q_c} = -M\ddot{q}_c + V'(q_c) = 0$$

$$S_E[q(\tau')] = \int_{-\infty}^{\infty} d\tau' \left\{ \frac{1}{2} M \left(\frac{dq}{d\tau'}\right)^2 + V(q) \right\}$$

$$q_c^{(0)}(\tau) = \frac{3\omega_0^2}{\lambda}\,\mathrm{sech}^2\,\frac{\omega_0\tau}{2}$$

$$\Gamma_0 = \frac{2\,|\mathrm{Im}E_0^{(0)}|}{\hbar} = \left(\frac{B_0}{2\pi\hbar M}\right)^{1/2}\left|\frac{\det(-M\partial_t^2 + M\omega_0^2)}{\det'(-M\partial_t^2 + V''(q_c^{(0)}))}\right|^{1/2} \exp -\frac{B_0}{\hbar}$$

$$B_0 = \int\limits_{-\infty}^{\infty}\left[\frac{1}{2}M\dot{q_c}^2 + V(q_c)\right]d\tau = \frac{36}{5}\frac{V_0}{\omega_0} \qquad\qquad V_0 = \frac{2}{3}\frac{M\omega_0^6}{\lambda^2}$$

$$\Gamma_0 = A_0 \exp -\frac{36}{5}\frac{V_0}{\hbar\omega_0} \qquad \text{with} \qquad A_0 = 6\omega_0\sqrt{\frac{6}{\pi}\frac{V_0}{\hbar\omega_0}}$$

Damped point particles

Minimum and saddle point of the full potential energy

$$(q, \mathbf{R}) = (0, \mathbf{0}) \qquad (q, \mathbf{R}) = (2\,\omega_0^2/\lambda, ..., 2\,C_k\omega_0^2/m_k\omega_k^2, ...)$$

multidimensional WKB is a possibility (hard) but using

$$\rho(x, \mathbf{R}; y, \mathbf{Q}, \beta) = \sum_n \psi_n(x, \mathbf{R})\psi_n^*(y, \mathbf{Q}) \exp -\beta E_n$$

$$\tilde{\rho}(x, y, \beta) = \int d\mathbf{R}\rho(x\mathbf{R}; y\mathbf{R}) = \int d\mathbf{R} \sum_n \psi_n(x, \mathbf{R})\psi_n^*(y, \mathbf{R}) \exp -\beta E_n$$

$$\tilde{\rho}(x, y, \beta) = \int d\mathbf{R}\langle x\mathbf{R}|e^{-\beta\mathcal{H}}|y\mathbf{R}\rangle \quad = \int d\mathbf{R} \int_{y,\mathbf{R}}^{x,\mathbf{R}} \mathcal{D}q(\tau')\mathcal{D}\mathbf{R}(\tau') \exp -\frac{S_E[q(\tau'), \mathbf{R}(\tau')]}{\hbar}$$

$$\int d\mathbf{R} |\psi_0(0,\mathbf{R})|^2 e^{-\tau E_0/\hbar} \approx \int d\mathbf{R} \int_{0,\mathbf{R}}^{0,\mathbf{R}} \mathcal{D}q(\tau')\mathcal{D}\mathbf{R}(\tau') \exp - \frac{S_E[q(\tau'),\mathbf{R}(\tau')]}{\hbar}$$

$$\tilde{\rho}(0,0,\beta) = \tilde{\rho}_0(\beta) \int_0^0 \mathcal{D}q(\tau') \exp - \frac{S_{eff}[q(\tau')]}{\hbar}$$

$$S_{eff}[q(\tau')] = \int_0^{\tau} \left\{ \frac{1}{2} M\dot{q}^2 + V(q) \right\} d\tau' + \frac{\eta}{4\pi} \int_{-\infty}^{\infty} d\tau'' \int_0^{\tau} d\tau' \frac{\{q(\tau') - q(\tau'')\}^2}{(\tau' - \tau'')^2}$$

$$\left. \frac{\delta S_{eff}}{\delta q} \right|_{q_c} = M\ddot{q}_c - \frac{\partial V}{\partial q_c} - \frac{\eta}{\pi} \int_{-\infty}^{\infty} d\tau'' \frac{[q_c(\tau') - q_c(\tau'')]}{(\tau' - \tau'')^2} = 0$$

$$\frac{d}{d\tau'} \left[\frac{1}{2} M \dot{q_c}^2 - V(q_c) \right] = \dot{q}_c(\tau') \frac{\eta}{\pi} \int_{-\infty}^{\infty} d\tau'' \frac{[q_c(\tau') - q_c(\tau'')]}{(\tau' - \tau'')^2}$$

Pre-factor: the eigenvalue problem $\hat{D}q(\tau') = \lambda q(\tau')$

$$\hat{D}q(\tau') = -M\frac{d^2q(\tau')}{d\tau'^2} + V''(q_c)q(\tau') + \frac{\eta}{\pi}\int_{-\infty}^{\infty} d\tau'' \frac{[q(\tau') - q(\tau'')]}{(\tau' - \tau'')^2}$$

$$\Gamma = A\exp-\frac{B}{\hbar} \qquad\qquad ||\dot{q}_c||^2 \neq B/M$$

$$A = \sqrt{\frac{||\dot{q}_c||^2}{2\pi\hbar}}\sqrt{\left|\frac{\det \hat{D}_0}{\det' \hat{D}}\right|} \qquad \text{with} \qquad ||\dot{q}_c||^2 \equiv \int_{-\infty}^{\infty} d\tau'\, \dot{q}_c^2(\tau')$$

$$B \equiv S_{eff}[q_c] = \int_{-\infty}^{\infty} \left\{\frac{1}{2}M\dot{q}_c^2 + V(q_c)\right\} d\tau' + \frac{\eta}{4\pi}\int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} d\tau'' \frac{\{q_c(\tau') - q_c(\tau'')\ \}^2}{(\tau' - \tau'')^2}$$

Weakly damped case $q_c(\tau') = q_c^{(0)}(\tau') + \eta\, q_c^{(1)}(\tau')$

Action correction $\Delta B \equiv B - B_0 = \frac{2\eta}{3\pi}\bar{q}^2 \int\limits_{-\infty}^{\infty} \frac{dx}{x^2}\left[\frac{\sinh^2 x - 3x\coth x + 3}{\sinh^2 x}\right]$

$$\Delta B = \frac{45}{\pi^3}\,\zeta(3)\,\alpha B_0 = \frac{12}{\pi^3}\,\zeta(3)\,\eta\,\bar{q}^2$$

$$\alpha \equiv \eta/2M\omega_0 = \gamma/\omega_0 \qquad\qquad \zeta(3) = \sum_{n=1}^{\infty} 1/n^3$$

Pre-factor correction $A \approx A_0(1 + c\ \alpha)$

$$\Gamma \approx \Gamma_0(1 + c\ \alpha)\left(1 - \frac{\Delta B}{\hbar}\right) \qquad\qquad \Delta B/\hbar \approx 1.7\alpha B_0/\hbar$$

$$B_0 \gg \hbar \qquad\qquad B_0 \gtrsim \hbar$$

Strongly damped case

$$q(\tau') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, q(\omega) e^{-i\omega\tau'}$$

Equation of motion

$$\left\{M\omega^2 + \eta|\omega| + M\omega_0^2\right\} q(\omega) - \frac{M\lambda}{4\pi} \int_{-\infty}^{\infty} d\omega'\, q(\omega - \omega')\, q(\omega') = 0$$

Ansatz for the overdamped solution $q_c(\omega) = A\, e^{-\kappa|\omega|}$

$$\eta|\omega| A\, e^{-\kappa|\omega|} + M\omega_0^2 A\, e^{-\kappa|\omega|} - \frac{3M\lambda A^2}{4\pi}\left|\frac{1}{\kappa} + |\omega|\right| e^{-\kappa|\omega|} = 0$$

↓

$$A = \frac{4\pi\eta}{3M\lambda} \quad \text{and} \quad \kappa = \frac{2\alpha}{\omega_0}$$

→

$$q_c(\tau') = \frac{4}{3} \frac{q_0}{1 + \left(\frac{\omega_0^2 \tau'}{2\gamma}\right)^2}$$

$$B = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{1}{2}\left(M\omega^2 + \eta|\omega| + M\omega_0^2\right) q_c^2(\omega) - \frac{M\lambda}{12\pi} \int_{-\infty}^{\infty} d\omega'\, q_c(\omega)\, q_c(\omega - \omega')\, q_c(\omega') \right\} d\omega$$

$$= \frac{2\pi}{9} \eta q_0^2 + \mathcal{O}\left(\frac{\omega_0}{\gamma}\right)$$

Pre-factor: the eigenvalue problem

$$A = \omega_0 \left(\frac{||\dot{\tilde{q}}_c||^2}{2\pi\hbar} \right)^{1/2} K^{1/2} \qquad K \equiv \left| \frac{\det \hat{H}_0}{\det'(\hat{H}_0 + \hat{V})} \right|$$

Frequency representation $\quad \langle t|\omega\rangle = (2\pi)^{-1/2} \exp -i\omega t$

$$\hat{H}_0 \psi_n(\omega) \equiv \left(1 + |\omega| + \frac{\omega^2}{4\alpha^2} \right) \psi_n(\omega) = \epsilon_0(\omega)\psi_n(\omega)$$

$$\hat{V}\psi_n(\omega) \equiv \int_{-\infty}^{+\infty} V(\omega - \omega' : \alpha)\psi_n(\omega')d\omega'$$

$$V(\omega) \equiv -\frac{3}{2\pi} \int_{-\infty}^{+\infty} dt\, \tilde{q}_c(t) \exp -i\omega t$$

$$\hat{H}_0 + \hat{V} \begin{cases} \lambda_0 \longrightarrow \psi_0(\omega) \\ \lambda_1 = 0 \longrightarrow \psi_1(\omega) \propto \omega V(\omega) \end{cases}$$

Evaluation of K $\quad \ln(\det \hat{O}) = \mathrm{tr}(\ln \hat{O})$

$$\ln K = \sum_{n=0}^{b} \langle \psi_n | \ln \hat{H}_0 | \psi_n \rangle + \sum_{n>b} \langle \psi_n | \ln \hat{H}_0 | \psi_n \rangle$$

$$- \sum_{n=0,\neq 1}^{b} \ln |\lambda_n| - \sum_{n>b} \langle \psi_n | \ln(\hat{H}_0 + \hat{V}) | \psi_n \rangle$$

$$\epsilon_0(\omega)\psi_n(\omega) + \int_{-\infty}^{+\infty} V(\omega,\omega')\, \psi_n(\omega')\, d\omega' = \lambda_n \psi_n(\omega)$$

$$\int_{-\infty}^{+\infty} \epsilon_0(\omega) |\psi_n(\omega)|^2 + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \psi_n^*(\omega) V(\omega,\omega')\, \psi_n(\omega')\, d\omega'\, d\omega = \lambda_n$$

$$\lambda_n = I_1^{(n)} + I_2^{(n)} \qquad I_2^{(n)} \ll I_1^{(n)}$$

$$\sum_{n>b} \ln\left(I_1^{(n)} + I_2^{(n)}\right) = \sum_{n>b} \ln I_1^{(n)} + \sum_{n>b} \ln\left(1 + \frac{I_2^{(n)}}{I_1^{(n)}}\right)$$

$$\approx \sum_{n>b} \ln I_1^{(n)} + \sum_{n>b} \left(\frac{I_2^{(n)}}{I_1^{(n)}}\right)$$

$$\ln K = \sum_{n=0}^{b} \langle\psi_n| \ln \hat{H}_0 |\psi_n\rangle - \sum_{n=0,\neq 1}^{b} \ln|\lambda_n| + \sum_{n=0}^{b} \langle\psi_n|\hat{H}_0^{-1}\hat{V}|\psi_n\rangle$$

$$+ \sum_{n>b} \left[\int_{-\infty}^{+\infty} \ln\epsilon_0(\omega)\, |\psi_n(\omega)|^2\, d\omega - \ln\left(\int_{-\infty}^{+\infty} \epsilon_0(\omega)\, |\psi_n(\omega)|^2\, d\omega\right)\right] - \mathrm{tr}(\hat{H}_0^{-1}\hat{V})$$

$$\mathrm{tr}\hat{H}_0^{-1}\hat{V} = V(0) \int_{-\infty}^{+\infty} \frac{d\omega}{1 + |\omega| + \omega^2/4\alpha^2} = -\frac{3}{\pi} f(\alpha) \int_{-\infty}^{+\infty} \tilde{q}_c(t')\, dt'$$

$$A = c\, A_0\, \alpha^{7/2}$$

$$c = \frac{4\sqrt{\pi}}{3}$$

$$f(\alpha) = \begin{cases} \frac{1}{\sqrt{1-\alpha^2}} \left(1 - \frac{2}{\pi} \tan^{-1} \frac{\alpha}{\sqrt{1-\alpha^2}}\right) & \text{se} \quad \alpha < 1 \\ \frac{1}{\sqrt{\alpha^2-1}} \frac{1}{\pi} \ln\left|\frac{\alpha+\sqrt{\alpha^2-1}}{\alpha-\sqrt{\alpha^2-1}}\right| & \text{se} \quad \alpha > 1 \end{cases}$$