Lista de exercícios 2:

Problemas 1-6: Questões 1, 2, 4, 6, 8 e 9 do Cap. II do Cohen-Tannoudji.

7) Usando a representação do operador P como

$$P = -i\hbar \frac{\partial}{\partial x},$$

mostre que

a) $\left[X, f(X)\right] = 0$

b) $\left[P, g(P)\right] = 0$

c) $\left[P, f(x)\right] = -i\hbar \dfrac{\partial f}{\partial x}$

d) $\left[P, X^k\right] = -i\hbar k X^{k-1}$

8)

Consider the states $|\psi\rangle = 9i\,|\phi_1\rangle + 2\,|\phi_2\rangle$ and $|\chi\rangle = -\frac{i}{\sqrt{2}}\,|\phi_1\rangle + \frac{1}{\sqrt{2}}\,|\phi_2\rangle$, where the two vectors $|\phi_1\rangle$ and $|\phi_2\rangle$ form a complete and orthonormal basis.

(a) Calculate the operators $|\psi\rangle\langle\chi|$ and $|\chi\rangle\langle\psi|$. Are they equal?

(b) Find the Hermitian conjugates of $|\psi\rangle$, $|\chi\rangle$, $|\psi\rangle\langle\chi|$, and $|\chi\rangle\langle\psi|$.

(c) Calculate $\mathrm{Tr}(|\psi\rangle\langle\chi|)$ and $\mathrm{Tr}(|\chi\rangle\langle\psi|)$. Are they equal?

(d) Calculate $|\psi\rangle\langle\psi|$ and $|\chi\rangle\langle\chi|$ and the traces $\mathrm{Tr}(|\psi\rangle\langle\psi|)$ and $\mathrm{Tr}(|\chi\rangle\langle\chi|)$. Are they projection operators?

9)

(a) Show that the sum of two projection operators cannot be a projection operator unless their product is zero.

(b) Show that the product of two projection operators cannot be a projection operator unless they commute.

10)

Consider the matrices $A = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & -i \\ 0 & i & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2i & 0 \\ i & 0 & -5i \end{pmatrix}$.

(a) Are A and B Hermitian? Calculate AB and BA and verify that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$; then calculate $[A,\ B]$ and verify that $\mathrm{Tr}([A,\ B]) = 0$.

(b) Find the eigenvalues and the normalized eigenvectors of A. Verify that the sum of the eigenvalues of A is equal to the value of $\mathrm{Tr}(A)$ calculated in (a) and that the three eigenvectors form a basis.

(c) Verify that $U^\dagger AU$ is diagonal and that $U^{-1} = U^\dagger$, where U is the matrix formed by the normalized eigenvectors of A.

(d) Calculate the inverse of $A' = U^\dagger AU$ and verify that A'^{-1} is a diagonal matrix whose eigenvalues are the inverse of those of A'.

11)

Consider a particle whose Hamiltonian matrix is $H = \begin{pmatrix} 2 & i & 0 \\ -i & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

(a) Is $\mid \lambda\rangle = \begin{pmatrix} i \\ 7i \\ -2 \end{pmatrix}$ an eigenstate of H? Is H Hermitian?

(b) Find the energy eigenvalues, a_1, a_2, and a_3, and the normalized energy eigenvectors, $\mid a_1\rangle$, $\mid a_2\rangle$, and $\mid a_3\rangle$, of H.

(c) Find the matrix corresponding to the operator obtained from the ket-bra product of the rst eigenvector $P = \mid a_1\rangle\langle a_1 \mid$. Is P a projection operator? Calculate the commutator $[P,\ H]$ rstly by using commutator algebra and then by using matrix products.

12)

Consider the matrices $A = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & i & 0 \\ 3 & 1 & 5 \\ 0 & -i & -2 \end{pmatrix}$.

(a) Check if A and B are Hermitian and nd the eigenvalues and eigenvectors of A. Any degeneracies?

(b) Verify that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, $\det(AB) = \det(A)\det(B)$, and $\det(B^\dagger) = (\det(B))^*$.

(c) Calculate the commutator $[A, B]$ and the anticommutator $\{A, B\}$.

(d) Calculate the inverses A^{-1}, B^{-1}, and $(AB)^{-1}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$.

13)

Consider the matrices $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(a) Find the eigenvalues and normalized eigenvectors of A and B. Denote the eigenvectors of A by $\mid a_1\rangle$, $\mid a_2\rangle$, $\mid a_3\rangle$ and those of B by $\mid b_1\rangle$, $\mid b_2\rangle$, $\mid b_3\rangle$. Are there any degenerate eigenvalues?

(b) Show that each of the sets $\mid a_1\rangle, \mid a_2\rangle, \mid a_3\rangle$ and $\mid b_1\rangle, \mid b_2\rangle, \mid b_3\rangle$ forms an orthonormal and complete basis, i.e., show that $\langle a_j \mid a_k\rangle = \delta_{jk}$ and $\sum_{j=1}^{3} \mid a_j\rangle\langle a_j \mid = I$, where I is the 3×3 unit matrix; then show that the same holds for $\mid b_1\rangle, \mid b_2\rangle, \mid b_3\rangle$.

(c) Find the matrix U of the transformation from the basis $\{\mid a\rangle\}$ to $\{\mid b\rangle\}$. Show that $U^{-1} = U^\dagger$. Verify that $U^\dagger U = I$. Calculate how the matrix A transforms under U, i.e., calculate $A' = UAU^\dagger$.