Cálculo variacional,

Ideia: determinar máximos/mínimos

p/ função $f = f(x)$: $\frac{df}{dx} = 0$ -> solução eq. = pto $x = x_0$

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

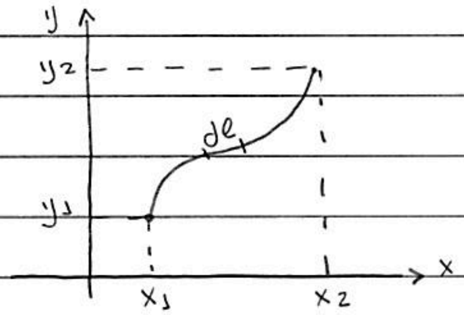
p/ funcional $f = f[y(x), y'(x), x]$: f é uma função das funções $y = y(x)$, $y' = y'(x)$ e da variável x.

nesse caso, defini-se

$$J \equiv \int_{x_0}^{x_1} f[y, y', x]\, dx$$

$\delta J = 0$ -> solução eq. = função $y = y(x)$!

Ex. 1: determinar curva $y = y(x)$ que corresponde à menor distância entre os ptos (x_1, y_1) e (x_2, y_2).



como $d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

↳ $dl = (dx^2 + dy^2 + dz^2)^{1/2}$

nesse caso, é necessário considerar:

$$L = \int_1^2 dl = \int_1^2 (dx^2 + dy^2)^{1/2}$$: comprimento curva $y = y(x)$!

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como $y = y(x)$, podemos considerar

$$d\ell = (dx^2 + dy^2)^{1/2} = dx\left(1 + (dy/dx)^2\right)^{1/2} = dx\left(1 + y'(x)\right)^{1/2}$$

$$\hookrightarrow L = \int_{x_1}^{x_2} dx \underbrace{\left(1 + y'^2(x)\right)^{1/2}}_{f[y']\ :\ \text{funcional}} \qquad (146.1.0)$$

Q.: Qual $y = y(x)$ tal que L é mínimo?

· caso geral:

$$J = \int_{x_1}^{x_2} dx\ f[y(x), y'(x), x] \quad ; \quad y = y(x) \qquad (146.1)$$

x → variável independente

$y(x)$ → " dependente

* problema: determinar $y = y(x)$ tal que J é um extremo (máximo/mínimo)

hipótese: $y = y(x)$ (a ser determinada) é extremo J;

definindo: $Y(\alpha, x) = y(x) + \alpha\,\eta(x)$: variação w.r.t. curva "ótima" $y = y(x)$

onde $\eta(x_1) = \eta(x_2) = 0$

(146.2)

$\hookrightarrow Y(\alpha, x_1) = y(x_1)$ e $Y(\alpha, x_2) = y(x_2)$

α: parâmetro

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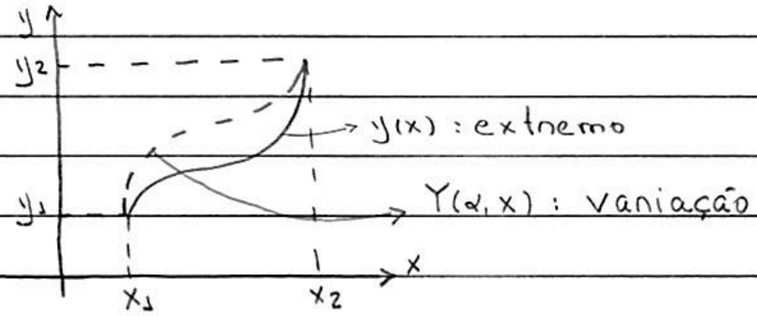


↳ Eq. (146.1) : $J = J(\alpha) = \int_{x_1}^{x_2} dx\, f\left[Y(\alpha,x), Y'(\alpha,x), x\right]$ (147.1)

* condição (necessária) p/ extremo : $\left.\dfrac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0$ (147.2)

↳ $\dfrac{\partial J}{\partial \alpha} = \dfrac{\partial}{\partial \alpha}\int_{x_1}^{x_2} dx\, f[Y, Y'; x] = \int_{x_1}^{x_2} dx \left(\dfrac{\partial f}{\partial Y}\underbrace{\dfrac{\partial Y}{\partial \alpha}}_{\eta} + \dfrac{\partial f}{\partial Y'}\underbrace{\dfrac{\partial Y'}{\partial \alpha}}_{\eta' = \frac{d\eta}{dx}} \right)$

pois os limites de integração são fixos !

$= \int_{x_1}^{x_2} \dfrac{\partial f}{\partial Y}\eta + \underbrace{\dfrac{\partial f}{\partial Y'}\dfrac{d\eta}{dx}} \, dx$

$-\dfrac{d}{dx}\left(\dfrac{\partial f}{\partial Y'}\right)\eta + \underbrace{\dfrac{d}{dx}\left(\eta \dfrac{\partial f}{\partial Y'}\right)}$

$= 0$, pois $\eta(x_1) = \eta(x_2) = 0$

$= \int_{x_1}^{x_2} dx \underbrace{\left[\dfrac{\partial f}{\partial Y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial Y'}\right)\right]}_{(I)} \eta(x)$ (147.3)

Como $\eta(x)$ é $\forall$, a condição $\left.\dfrac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0$

↳ $(I) = 0$ e $Y(0,x) = y(x)$, i.e.,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$: Equação de Euler-Lagrange (148.1)

Resumo: determinação extremo Eq. (146.1) ~ determinação $y = y(x)$ a partir Eq. (148.1)

Ex. 1: continuação,

nesse caso, o funcional f é dado pela Eq. (146.1.0),

$$f[y'] = \left(1 + y'^2(x)\right)^{1/2}$$

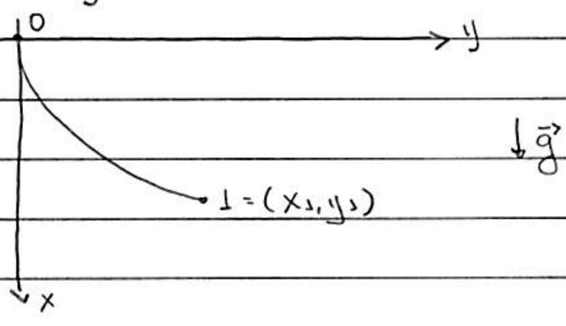
↳ $\frac{\partial f}{\partial y} = 0$ e $\frac{\partial f}{\partial y'} = \frac{y'}{(1+y'^2)^{1/2}}$

Eq. (148.1): $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \rightarrow \frac{y'}{(1+y'^2)^{1/2}} = C = cte$

↳ $y = ax + b$

Obs.: determinação a, b ~ ptos inicial (x_1, y_1) e final (x_2, y_2).

Ex. 2: Ex. 6.2, Marion: considerar partícula massa m sob $\vec{F} = m\vec{g}$, movimento plano xy entre ptos (x_0, y_0) e (x_1, y_1) ⊕ condição inicial $\vec{v}(t_0) = 0$; determinar trajetória correspondente ao menor $\Delta t = t_1 - t_0$.



hipótese: $(x_0, y_0) = (0, 0)$

nesse caso: $\Delta t = t_1 - t_0 = \int_0^1 \frac{dl}{v}$

como: $dl = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + y'^2}$; $y = y(x)$ (149.1)

$y' = \frac{dy}{dx}$

conservação energia total $\to$ $v = \sqrt{2gx}$ (149.2)
(veja abaixo)

$\hookrightarrow \Delta t = t_1 - t_0 = \frac{1}{\sqrt{2g}} \int_0^{x_1} dx \underbrace{\sqrt{\frac{1+y'^2}{x}}}_{= f[y', x]}$: notar a forma do funcional!

como:

$\frac{\partial f}{\partial y} = 0$ e $\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{x(1+y'^2)}}$

$\hookrightarrow$ Eq. (148.1) : $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \to \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{x(1+y'^2)}} = \text{cte} = \frac{1}{\sqrt{2a}}$

exercício $\to$ $y'^2 = \frac{x^2}{(2a - x)x}$ $\to$ $y = \int dx \frac{x}{\sqrt{2ax - x^2}}$

se $x = a(1 - \cos\theta)$

exercício $\to$ $y = \int d\theta\, a(1 - \cos\theta) = a(\theta - \sin\theta) + C_1$

$\hookrightarrow$ $x = x(\theta) = a(1 - \cos\theta)$: eq. paramétrica
$y = y(\theta) = a(\theta - \sin\theta) + C_1$ da trajetória

Lembrar: determinação ctes C_1 e a ~ p/os $(x_0, y_0) = 0$ e (x_1, y_1)

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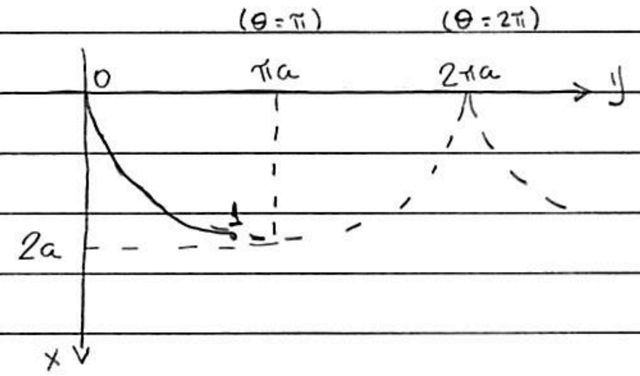
notar: $\theta = 0 \to x = 0 \to y = 0$: $y(0) = 0 \to C_1 = 0$!

↳ $x = x(\theta) = a(1 - \cos\theta)$

$y = y(\theta) = a(\theta - \sin\theta)$: eq. paramétrica ciclóide



Obs.:

(i) brachistochrone problem ; brachistos : shortest
chronos : time

(ii) Eq. (149.1) : a escolha $y = y(x)$ não é única,
veja exemplo geodésica abaixo

(iii) Eq. (149.2) :

$$F_x = mg \to U(x) - U(0) = -\int_0^x mg\,dx' = -mgx$$

↳ $E = U(x=0) = 0 = T + U(x) = \frac{1}{2}mv^2 - mgx = 0 \to v = \sqrt{2gx}$

· Alternativa p/ a determinação de $y = y(x)$: interessante p/ funcionais da forma $f = f[y, y']$.

inicial : considerar caso geral $f = f[y, y', x]$

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$\hookrightarrow \dfrac{df}{dx} = \dfrac{\partial f}{\partial y}\dfrac{dy}{dx} + \dfrac{\partial f}{\partial y'}\dfrac{dy'}{dx} + \dfrac{\partial f}{\partial x}$

$$= \frac{\partial f}{\partial y} y' + \underbrace{\frac{\partial f}{\partial y'} y''}_{\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) - y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)} + \frac{\partial f}{\partial x}$$

$\hookrightarrow \dfrac{df}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{d}{dx}\left(y'\dfrac{\partial f}{\partial y'}\right) + y'\underbrace{\left(\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right)\right)}_{= 0,\ \text{Eq. (148.1)}}$

$\hookrightarrow \dfrac{\partial f}{\partial x} - \dfrac{d}{dx}\left(f - y'\dfrac{\partial f}{\partial y'}\right) = 0$

notar: se $f = f[y, y'] \rightarrow \partial f/\partial x = 0$

$$\hookrightarrow f - y'\frac{\partial f}{\partial y'} = cte \qquad (151.1)$$

Ex. 3: Ex. 6.4, Marion: determinar curva que corresponde à menor distância entre os ptos P_1 e P_2;
curva $\subset$ superfície: geodésica.

hipótese: superfície = esfera raio $\underline{\underline{a}}$.

nesse caso, é necessário considerar:

$$L = \int_1^2 d\ell \qquad (151.2)$$

simetria sistema $\rightarrow$ $d\ell$ em coordenadas esféricas!

$d\vec{\ell} = dr\hat{r} + r d\theta\,\hat{\theta} + r\sin\theta\, d\varphi\,\hat{\varphi}$: pg. 50.

↳ $d\ell = \left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2\right)^{1/2}$

como $r = a = cte$ → $d\ell = \left(a^2 d\theta^2 + a^2\sin^2\theta\, d\varphi^2\right)^{1/2}$

considerando a possível escolha $\theta = \theta(\varphi)$, i.e., φ é a variável independente

↳ $d\ell = a\, d\varphi\left(\theta'^2 + \sin^2\theta\right)^{1/2}$; $\theta' = d\theta/d\varphi$

Eq. (151.2) :

$$L = a\int_1^2 d\varphi\, \underbrace{\left(\theta'^2 + \sin^2\theta\right)^{1/2}}_{f[\theta,\theta']}$$

↳ $\dfrac{\partial f}{\partial\theta} = \dfrac{\sin\theta\cos\theta}{\left(\theta'^2 + \sin^2\theta\right)^{1/2}}$ e $\dfrac{\partial f}{\partial\theta'} = \dfrac{\theta'}{\left(\theta'^2 + \sin^2\theta\right)^{1/2}}$

Eq. (148.1) : $\dfrac{\sin\theta\cos\theta}{\left(\theta'^2 + \sin^2\theta\right)^{1/2}} - \dfrac{d}{d\varphi}\left(\dfrac{\theta'}{\left(\theta'^2 + \sin^2\theta\right)^{1/2}}\right) = 0$

· alternativa determinação $\theta = \theta(\varphi)$: via Eq. (151.1) ;

$$f - y'\frac{\partial f}{\partial y'} = \left(\theta'^2 + \sin^2\theta\right)^{1/2} - \frac{\theta'^2}{\left(\theta'^2 + \sin^2\theta\right)^{1/2}} = cte \equiv b$$

↳ exercício → $b\theta' = b\dfrac{d\theta}{d\varphi} = \sin\theta\left(\sin^2\theta - b^2\right)$

↳ $\dfrac{d\varphi}{d\theta} = \dfrac{b}{\sin\theta\left(\sin^2\theta - b^2\right)^{1/2}} = \dfrac{b\csc^2\theta}{\left(1 - b^2\csc^2\theta\right)^{1/2}}$ (152.1)

$$\varphi = \int d\theta \, \frac{b \csc^2\theta}{(1 - b^2\csc^2\theta)^{1/2}}$$

exercício : mostrar que

$$\varphi = -\frac{b}{2}\int \frac{dx}{(-b^2x^2 + (1+b^2)x - 1)^{1/2}} \quad ; \quad t = \frac{1}{\sin\theta} \quad e \quad x = t^2$$

$$= \frac{1}{2}\sin^{-1}\left(-2\,\frac{b^2}{1-b^2}\,\frac{1}{\sin^2\theta} + \frac{(1+b^2)}{(1-b^2)} \right) + C$$

se $C = 2\alpha - \pi/2$ e $\beta^2 = \dfrac{1-b^2}{b^2}$

↳ $\beta \sin(\varphi - \alpha) = \cot\theta$ (153.1)

notar: Eq. (153.1) pode ser escrita como:

$$\underbrace{\beta\cos\alpha}_{\equiv A}\,(\underbrace{a\sin\varphi\sin\theta}_{y}) - \underbrace{\beta\sin\alpha}_{\equiv B}\,(\underbrace{a\sin\theta\cos\varphi}_{x}) = \underbrace{a\cos\theta}_{z}$$

↳ $Ay - Bx = z$: eq. plano;
origem O (centro da esfera) ∈ plano

↳ geodésica = intersecção plano / esfera = "grande círculo"

Obs.: cte A e B ~ ptos P_1 e P_2 !

· alternativa via Eq. (151.1):

considerando $\varphi = \varphi(\theta)$, temos que

$$d\ell = a\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)^{1/2} = a\,d\theta\left(1 + \sin^2\theta\,\varphi'^2\right)^{1/2} \quad ; \quad \varphi' = \frac{d\varphi}{d\theta}$$

↳ Eq. (151.2): $L = a\int_1^2 d\theta\, \underbrace{\left(1 + \sin^2\theta\,\varphi'^2\right)^{1/2}}_{f[\varphi', \theta]}$

→ variável independente!

↳ $\dfrac{\partial f}{\partial \varphi} = 0$ e $\dfrac{\partial f}{\partial \varphi'} = \dfrac{\varphi'}{\left(1 + \sin^2\theta\,\varphi'^2\right)^{1/2}} \sin^2\theta$

Eq. (148.1): $\dfrac{d}{d\theta}\left(\dfrac{\partial f}{\partial \varphi'}\right) = 0 \rightarrow \dfrac{\partial f}{\partial \varphi'} = cte \equiv b$

↳ $$\frac{\sin^2\theta\,\varphi'}{\left(1 + \sin^2\theta\,\varphi'^2\right)^{1/2}} = b \quad \text{exercício} \rightarrow \text{Eq. (152.1)} \qquad (154.1)$$

- solução Eq. (152.1)

hipótese: pto $P_1 = (0, 0, a)$: $r = a$ e $\theta = 0$

notar: $\left.\dfrac{d\varphi}{d\theta}\right|_{\theta=0} \rightarrow +\infty$

p/ $\varphi' < +\infty$ a cte b deve ser nula

$b = 0 \rightarrow \varphi' = 0 \rightarrow \varphi = cte$: equação plano;

origem $O \in$ plano!

· considerar: f funcional de várias variáveis dependentes $y_i = y_i(x)$.

inicial: caso particular

$$f = f[y_1(x), y'_1(x), y_2(x), y'_2(x), x] \qquad (155.1)$$

nesse caso, Eq. (146.1):

$$J = \int_{x_1}^{x_2} dx\, f[y_1, y'_1, y_2, y'_2, x] \qquad (155.2)$$

de modo análogo ao caso $f = f[y, y', x]$, define-se

$$Y_1(\alpha, x) = y_1(x) + \alpha\eta_1(x) \quad ; \quad \eta_1(x_1) = \eta_1(x_2) = 0$$

$$Y_2(\alpha, x) + y_2(x) + \alpha\eta_2(x) \quad ; \quad \eta_2(x_1) = \eta_2(x_2) = 0$$

↳ $$\frac{\partial J}{\partial\alpha} = \int_{x_1}^{x_2} dx \left[\frac{\partial f}{\partial Y_1} - \frac{d}{dx}\left(\frac{\partial f}{\partial Y'_1}\right)\right]\eta_1 + \left[\frac{\partial f}{\partial Y_2} - \frac{d}{dx}\left(\frac{\partial f}{\partial Y'_2}\right)\right]\eta_2$$

como $\eta_1(x)$ e $\eta_2(x)$ são independentes a condição

→ $$\left.\frac{\partial J}{\partial\alpha}\right|_{\alpha=0} = 0$$

↳ $$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'_i}\right) = 0 \quad ; \quad i = 1, 2 \qquad (155.3)$$

caso geral: $f = f[y_1, y_2, \dots y_n, y'_1, y'_2 \dots y'_n, x] = f[y_i, y'_i, x]$

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$$\hookrightarrow \frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'_i}\right) = 0 \quad ; \quad i = 1, 2, \ldots, n \qquad (156.1)$$

Ex. 1 : solução alternativa,

considerar $x = x(t)$: eq. paramétrica curva
$y = y(t)$

$$\hookrightarrow d\ell = (dx^2 + dy^2)^{1/2} \cdot \frac{dt}{dt} = dt\,(x'^2 + y'^2)^{1/2} \quad ; \quad x' = dx/dt$$
$$y' = dy/dt$$

$$L = \int_1^2 d\ell = \int_{t_1}^{t_2} dt\, \underbrace{(x'^2 + y'^2)^{1/2}}_{f[x', y']}$$

como $\dfrac{\partial f}{\partial x} = 0 \qquad \dfrac{\partial f}{\partial x'} = \dfrac{x'}{(x'^2 + y'^2)^{1/2}}$

$$\frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial y'} = \frac{y'}{(x'^2 + y'^2)^{1/2}}$$

Eq. (156.1) : $\dfrac{d}{dt}\left(\dfrac{\partial f}{\partial x'}\right) = 0 \rightarrow \dfrac{x'}{(x'^2 + y'^2)^{1/2}} = C_1$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial y'}\right) = 0 \rightarrow \frac{y'}{(x'^2 + y'^2)^{1/2}} = C_2$$

$$\hookrightarrow \frac{y'}{x'} = cte = a = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dx} = a$$

$$\hookrightarrow y = ax + b.$$

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· Determinação extremo sob vínculo,

considerar $f = f[y(x), y'(x), z(x), z'(x), x]$

e $g[y, z, x] = 0$: eq. do vínculo (157.1)

↳ ∃ relação entre as funções $y = y(x)$ e $z = z(x)$!

Ex.: geodésica,

como curva ⊂ esfera raio a

↳ $x^2 + y^2 + z^2 = a^2$ → $g = x^2 + y^2 + z^2 - a^2 = 0$

ou $r = a$ → $g = r - a = 0$

Eq. (155.2) :

$$J = \int_{x_1}^{x_2} dx\, f[y, y', z, z', x]$$

verifica-se que (veja abaixo)

$$\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0 \rightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \lambda(x)\frac{\partial g}{\partial y} = 0$$

(157.2)

$$\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) + \lambda(x)\frac{\partial g}{\partial z} = 0$$

↳ multiplicador de Lagrange

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notar:

Eq. (157.2) ⊕ vínculo : 3 equações

3 funções a determinar: $y = y(x)$, $Z = Z(x)$, $\lambda = \lambda(x)$.

· caso geral:

$$f = f[y_i, y'_i, x] \quad ; \quad i = 1, 2, \ldots, n$$

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'_i}\right) + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial y_i} = 0$$

(158.1)

$$g_j[y_i, x] = 0 \quad , \quad j = 1, 2, \ldots, m$$

· vamos verificar Eq. (157.2),

de modo análogo ao caso particular (155.1), define-se

$$Y(\alpha, x) = y(x) + \alpha \eta_1(x) \quad ; \quad \eta_1(x_1) = \eta_1(x_2) = 0$$

$$Z(\alpha, x) = Z(x) + \alpha \eta_2(x) \quad ; \quad \eta_2(x_1) = \eta_2(x_2) = 0$$

$$\hookrightarrow \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} dx \left[\frac{\partial f}{\partial Y} - \frac{d}{dx}\left(\frac{\partial f}{\partial Y'}\right)\right]\eta_1 + \left[\frac{\partial f}{\partial Z} - \frac{d}{dx}\left(\frac{\partial f}{\partial Z'}\right)\right]\eta_2 \quad (158.2)$$

* ≠ caso anterior, $\eta_1(x)$ e $\eta_2(x)$ não são independentes devido ao vínculo (157.1)!

Como: $\eta_1 = \frac{\partial Y}{\partial \alpha}$; $\eta_2 = \frac{\partial Z}{\partial \alpha}$

$$dg = \frac{\partial g}{\partial Y}\frac{\partial Y}{\partial \alpha} + \frac{\partial g}{\partial Z}\frac{\partial Z}{\partial \alpha} = 0 \quad \rightarrow \quad \eta_2 = -\eta_1 \frac{\partial g}{\partial Y}\left(\frac{\partial g}{\partial Z}\right)^{-1}$$

(158.3)

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Eq. (158.3) em (158.2) :

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} dx \left[\underbrace{\frac{\partial f}{\partial Y} - \frac{d}{dx}\left(\frac{\partial f}{\partial Y'}\right)}_{(\mathrm{I})} - \underbrace{\left(\frac{\partial f}{\partial Z} - \frac{d}{dx}\left(\frac{\partial f}{\partial Z'}\right)\right)}_{(\mathrm{II})} \frac{\partial g}{\partial Y}\left(\frac{\partial g}{\partial Z}\right)^{-1} \right] \eta_1$$

nesse caso

$$\frac{\partial J}{\partial \alpha} = 0 \to (\mathrm{I}) - (\mathrm{II}) \frac{\partial g}{\partial Y}\left(\frac{\partial g}{\partial Z}\right)^{-1} = 0$$

$$\hookrightarrow (\mathrm{I}) \left(\frac{\partial g}{\partial Y}\right)^{-1} = (\mathrm{II}) \left(\frac{\partial g}{\partial Z}\right)^{-1} \equiv -\lambda(x) \qquad (159.1)$$

pois L.H.S. função apenas y e
R.H.S. " " z !

$\alpha = 0$ em (159.1) $\to$ (157.2) !

Ex. 4 : P 6.12, Marion : determinar a geodésica p/ a esfera, porém considerando explicitamente o vínculo $r = a$ = raio da esfera.

Ex. 3 : $L = \int_1^2 dl$

$$dl = \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2\right)^{1/2}$$

considerando $r = r(\theta)$ e $\varphi = \varphi(\theta)$:

$$L = \int_1^2 d\theta \underbrace{\left(r^2 + r'^2 + r^2 \sin^2\theta \, \varphi'^2\right)^{1/2}}_{f = f[r, r', \varphi', \theta]} \quad ; \quad r' = dr/d\theta, \quad \varphi' = d\varphi/d\theta$$

$\hookrightarrow$ variável independente !

e $g[r] = r - a = 0$: eq. do vínculo

como : $\dfrac{\partial f}{\partial r} = \dfrac{r(1+\sin^2\theta\ \varphi'^2)}{(r^2+r'^2+r^2\sin^2\theta\,\varphi'^2)^{1/2}}$; $\dfrac{\partial f}{\partial r'} = \dfrac{r'}{(r^2+r'^2+r^2\sin^2\theta\,\varphi'^2)^{1/2}}$

$$\frac{\partial f}{\partial \varphi} = 0 \qquad \frac{\partial f}{\partial \varphi'} = \frac{r^2\sin^2\theta\,\varphi'}{(r^2+r'^2+r^2\sin^2\theta\,\varphi'^2)^{1/2}}$$

$$\frac{\partial g}{\partial r} = 1 \qquad \frac{\partial g}{\partial \varphi} = 0$$

Eq. (158.1):

$$r: \quad \frac{r(1+\sin^2\theta\ \varphi'^2)}{(r^2+r'^2+r^2\sin^2\theta\,\varphi'^2)^{1/2}} - \frac{d}{d\theta}\left(\frac{r'}{(r^2+r'^2+r^2\sin^2\theta\,\varphi'^2)^{1/2}}\right) + \lambda = 0 \qquad (160.1)$$

$$\varphi: \quad \frac{r^2\sin^2\theta\,\varphi'}{(r^2+r'^2+r^2\sin^2\theta\,\varphi'^2)^{1/2}} = \text{cte} \qquad (160.2)$$

↳ $g = r - a = 0$ em (160.2):

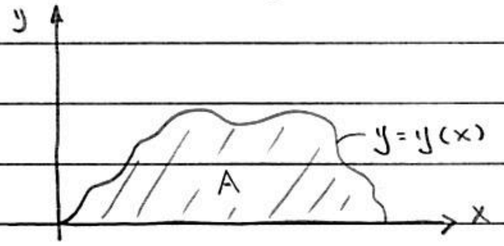
$$\frac{\sin^2\theta\,\varphi'}{(1+\sin^2\theta\,\varphi'^2)^{1/2}} = \text{cte} \ : \text{Eq. (154.1)} \rightarrow \varphi = \text{cte}$$

↳ $g = r - a = 0$ e $\varphi = \text{cte}$ em (160.1):

$$\lambda(\theta) = -(1+\sin^2\theta\,\varphi'^2)^{1/2}$$

Obs.: $\lambda \sim$ força de vínculo na mecânica Lagrangiana (veja, cap. 7!).

Ex. 5 : Ex. 6.6, Marion : considerar corda comprimento $\underline{\underline{L}}$, extremidade fixa na origem O; determinar curva $y = y(x)$ tal que a área A seja máxima.



nesse caso : $$A = \int_0^{x_1} y\,dx \qquad (161.1)$$

como : $d\ell^2 = dx^2 + dy^2$ e escolhendo $\underline{\underline{y}}$ como variável dependente

$$\hookrightarrow dx = \left(d\ell^2 - dy^2\right)^{1/2} = d\ell\left(1 - y'^2\right)^{1/2} \quad ; \quad y = y(\ell), \quad y' = dy/d\ell$$

Eq. (161.1) : $$A = \int_0^{\ell} d\ell\, \underbrace{y\left(1 - y'^2\right)^{1/2}}_{f = f[y, y']}$$

como $df/d\ell = 0$, vamos considerar Eq. (151.1) :

$$f - y'\frac{df}{dy'} = y\left(1 - y'^2\right)^{1/2} - \frac{y'y\,(1/2)\,(-2y')}{\left(1 - y'^2\right)^{1/2}} = a = cte$$

$$\hookrightarrow y(1 - \cancel{y'^2}) - \cancel{yy'^2} = a\left(1 - y'^2\right)^{1/2} \longrightarrow y' = \frac{dy}{d\ell} = \sqrt{1 - y^2/a^2}$$

$$\hookrightarrow d\ell = \frac{dy}{\sqrt{1 - y^2/a^2}} \longrightarrow a\sin^{-1}(y/a) = \ell + cte$$

$$\hookrightarrow y = a\sin(\ell/a) + b$$

como $d\ell^2 = dx^2 + dy^2 \rightarrow 1 = \left(\frac{dx}{d\ell}\right)^2 + \left(\frac{dy}{d\ell}\right)^2$

$$1 = \left(\frac{dx}{d\ell}\right)^2 + \cos^2\frac{\ell}{a} \rightarrow \frac{dx}{d\ell} = \pm \sin\frac{\ell}{a}$$

$\hookrightarrow x = +a\cos\frac{\ell}{a} + c$

$\hookrightarrow x = x(\ell) = a\cos\frac{\ell}{a} + c$

: eq. paramétrica $y = y(x)$.

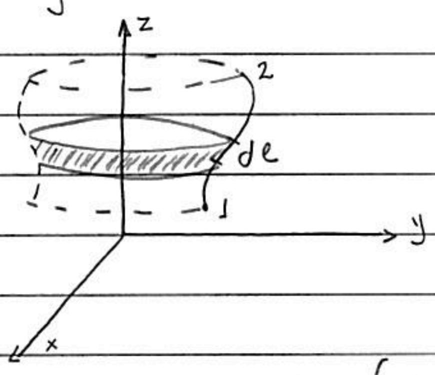
$y = y(\ell) = a\sin\frac{\ell}{a} + b$

como $(x-c)^2 + (y-b)^2 = a^2$: semi-círculo : raio a centro (c, b)

nesse caso: $\pi a = L$

$b = 0$ e $c = a = L/\pi$.

Ex. 6: Ex. 6.3, Marion: determinar curva $z = z(y)$ tal que a área da superfície de revolução (em torno eixo $\hat{z}$) seja mínima.



curva $\in$ plano yz

$P_1 = (0, y_1, z_1)$ e $P_2 = (0, y_2, z_2)$: fixos

nesse caso: $A = \int dA = \int 2\pi y \, d\ell$

como: $d\ell = (dy^2 + dz^2)$ e considerando $z = z(y)$

$\hookrightarrow dA = 2\pi y \, dy \, (1 + z'^2)^{1/2} \quad ; \quad z' = dz/dy$

$\hookrightarrow$ $A = 2\pi \int_{y_1}^{y_2} dy \; \underbrace{y(1+z'^2)^{1/2}}_{f = f[z', y]}$ → variável independente.

como $\dfrac{\partial f}{\partial z} = 0$ e $\dfrac{\partial f}{\partial z'} = \dfrac{y z'}{\sqrt{1+z'^2}}$

Eq. (148.1) : $\dfrac{d}{dy}\left(\dfrac{\partial f}{\partial z'}\right) = 0 \rightarrow \dfrac{y z'}{\sqrt{1+z'^2}} = a = cte$

$\hookrightarrow$ $z' = \dfrac{dz}{dy} = \dfrac{a}{\sqrt{y^2 - a^2}}$ $\xrightarrow{\text{exercício}}$ $z = a \cosh^{-1}(y/a) + b$

ou $y = a \cosh\left(\dfrac{z-b}{a}\right)$: catenária

Lembrar: ctes $\underline{\underline{a}}$ e $\underline{\underline{b}}$ ~ ptos fixos $P_1 = (0, y_1, z_1)$ e $P_2 = (0, y_2, z_2)$.

· sobre a notação δ;

Eq. (147.3) :
$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial Y} - \frac{d}{dx}\left(\frac{\partial f}{\partial Y}\right)\right) \frac{\partial Y}{\partial \alpha} \qquad (163.1)$$

Eq. (163.1) pode ser escrita como

$$\frac{\partial J}{\partial \alpha} d\alpha = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial Y} - \frac{d}{dx}\left(\frac{\partial f}{\partial Y}\right)\right) \frac{\partial Y}{\partial \alpha} d\alpha$$

definindo : $\delta J \equiv \dfrac{\partial J}{\partial \alpha} d\alpha$

$\delta Y = \dfrac{\partial Y}{\partial \alpha} d\alpha$

/ /

$$\hookrightarrow \quad \delta J = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right) \delta y$$

vamos determinar Eq. (148.1) utilizando a notação δ,

$$\delta J = \delta \int_{x_1}^{x_2} dx\, f[y, y', x] = \int_{x_1}^{x_2} dx\, \delta f$$

→ limites de integração fixos

$$= \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' \right) \qquad (164.1)$$

como $\delta y' = \delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}\delta y$ em (164.1) ⊕ integração por partes

$$\hookrightarrow \quad \delta J = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right) \delta y$$

a condição $\delta J = 0$ → Eq. (148.1)