Formulação Lagrangiana e Hamiltoniana.

ideia: introdução formalismos Lagrangiano e Hamiltoniano p/
(i) partícula carga q sob campos EM;
(ii) campo EM.

inicial: revisão mecânica clássica. • Ref.: Secs. 12.1 - 12.3, 12.7 e 12.10, Jackson

(1) formalismo Lagrangiano.

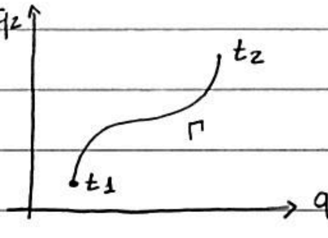
considerar: sistema N-partículas massas m_i, descritas pelas coordenadas generalizadas

$q_1, q_2, \ldots, q_s$, $s = 3N - K$
↳ # vínculos!

estado do sistema = pto no espaço de configurações (espaço dimensão = s, definido pelas coordenadas generalizadas)

↳ p/ intervalo $\Delta t = t_2 - t_1$: movimento sistema = = trajetória (curva Γ) no espaço de config.

Ex.: sistema descrito pelas coord. generalizadas q_1 e q_2.



mecânica Lagrangiana: baseada no princípio de Hamilton:

o movimento de um sistema no intervalo de tempo $\Delta t = t_2 - t_1$ é tal que a ação S,

$$S = \int_{t_1}^{t_2} dt\, L(q_j, \dot{q}_j, t) \qquad (221.1)$$

é estacionária ($\delta S = 0$) para a trajetória do sistema.

Aqui, $L = T - U$: função Lagrangiana (221.2)

$q_j = q_j(t)$
e $\dot{q}_j = \dot{q}_j(t)$: velocidades generalizadas

em particular, p/ sistemas conservativos, temos que

$$L = \frac{1}{2} \sum_j m_j \dot{q}_j^2 - U(q_1, q_2, \ldots, q_s) \qquad (221.3)$$

verifica-se que (veja abaixo),

$$\delta S = 0 \rightarrow \frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \quad : \text{Eq. de Lagrange} \qquad (221.4)$$

$j = 1, 2, \ldots, s = 3N - k$

considerar : coord. generalizadas = coord. cartesianas,

↳ Eq. (221.3) : $L = \frac{1}{2} \sum_{i=1}^{N} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) - U(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$

↳ $\frac{\partial L}{\partial x_i} = -\frac{\partial U}{\partial x_i}$ e $\frac{\partial L}{\partial \dot{x}_i} = m_i \dot{x}_i$

Eq. (221.4) : $-\frac{\partial U}{\partial x_i} - m_i \ddot{x}_i = 0 \rightarrow m_i \ddot{x}_i = -\frac{\partial U}{\partial x_i}$

e, similar : $m_i \ddot{y}_i = -\frac{\partial U}{\partial y_i}$; $m_i \ddot{z}_i = -\frac{\partial U}{\partial z_i}$: eqs. de movimento : 2ª Lei Newton

· Vamos verificar Eq. (221.4),

considerar: variação $q_j(t) \to q_j(t) + \delta q_j(t)$ da trajetória sistema w.r.t. trajetória $q_j(t)$ que corresponde extremo ação S ⊕ condições : $\delta q_j(t_1) = \delta q_j(t_2) = 0$ (*)

$$\hookrightarrow \delta S = \int_{t_1}^{t_2} dt\ \delta L = \int dt\ L\left[q_j(t) + \delta q_j(t)\ ;\ \dot{q}_j(t) + \delta \dot{q}_j(t)\ ;\ t\right] -$$

$$- L\left[q_j(t)\ ;\ \dot{q}_j(t)\ ;\ t\right]$$

como, em 1ª ordem em δq_j,

$$L(q_j + \delta q_j, \dot{q}_j + \delta \dot{q}_j, t) \simeq L(q_j, \dot{q}_j, t) + \frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j.$$

temos que,

$$\delta S = \int_{t_1}^{t_2} dt\ \frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j$$

como $\delta \dot{q}_j = \delta\left(\frac{dq_j}{dt}\right) = \frac{d}{dt}(\delta q_j)$

$$\hookrightarrow \delta S = \int_{t_1}^{t_2} dt\ \frac{\partial L}{\partial q_j}\delta q_j + \underbrace{\frac{\partial L}{\partial \dot{q}_j}\frac{d}{dt}\delta q_j}_{-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \underbrace{\frac{\partial L}{\partial \dot{q}_j}\delta q_j(t)\Big|_{t_1}^{t_2}}_{=0\ \sim\ (*)}}$$

$$\hookrightarrow \delta S = \int_{t_1}^{t_2} dt\left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right)\delta q_j = 0$$

como $\delta q_j(t)$ é arbitrário $\to$ Eq. (221.4)!

notar: $\frac{\partial L}{\partial \dot{x}_i} = m_i \dot{x}_i = p_{x,i}$: componente x, momento linear partícula m_i (222.1)

↳ generalização (222.1) p/ $L = L(q_j, \dot{q}_j, t)$:

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$: momento generalizado ou momento canonicamente conjugado (222.2)

(2) formalismo Hamiltoniano.

Lembrar definição função Hamiltoniana, sistema c/ n graus de liberdade:

$$H(p,q,t) = \sum_{i=1}^{n} \dot{q}_j p_j - L(q,\dot{q},t) \qquad (222.3)$$

onde : $q = (q_1, q_2, \ldots, q_n)$: 2n variáveis independentes!
$p = (p_1, p_2, \ldots, p_n)$

Lembrar: - Eq. (222.3) : transformada de Legendre variáveis $(q, \dot{q}, t) \rightarrow (q, p, t)$;
- q_j e p_j : variáveis canonicamente conjugadas.

eqs. de movimento:

$$\left. \begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial p_j} \\ \dot{p}_j &= -\frac{\partial H}{\partial q_j} \end{aligned} \right\}$$ 2n eqs. diferenciais 1ª ordem: Eqs. de Hamilton (eqs. canônicas de movimento) (222.4)

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

Eq. (222.4)

notar: $$\frac{dH}{dt} = \frac{\partial H}{\partial q_j}\dot{q}_j + \frac{\partial H}{\partial p_j}\dot{p}_j + \frac{\partial H}{\partial t} = -\dot{p}_j\dot{q}_j + \dot{q}_j\dot{p}_j + \frac{\partial H}{\partial t}$$

$\hookrightarrow \frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$:

- H depende t̲ apenas explicitamente NOT implicitamente via q_j e p_j ;
- Se $L = L(q_j, \dot{q}_j) \to \frac{dH}{dt} = 0 \to H$: cte de movimento !

· Formulação relativística : partícula sob campo EM.

considerar : partícula massa m, carga q sob campos $\vec{E}$ e $\vec{B}$.

Lembrar : (i) Lagrangiana, limite não-relativístico.

$$L = \frac{1}{2} m u^2 - q\Phi + \frac{1}{c} q \vec{u} \cdot \vec{A} \qquad (223.1)$$

velocidade partícula

(ii) eq. de movimento partícula (215.2), forma covariante.

$$\frac{dP^\alpha}{d\tau} = \frac{1}{c} q F^{\alpha\beta} U_\beta \quad ; \quad P^\alpha = (\mathcal{E}/c, \vec{p}) \ \text{e} \ U^\alpha = (\gamma c, \gamma \vec{u}) \qquad (223.2)$$

ou componentes espacial e temporal :

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{1}{c} \vec{u} \times \vec{B}\right)$$

$$\frac{d\mathcal{E}}{dt} = q \vec{u} \cdot \vec{E}$$

ideia: determinar Lagrangiana relativística tal que
- eq. de movimento ~ Eqs. (223.2) e
- lim $v/c \ll 1$ ~ Eq. (223.1) !

· sobre a Lagrangiana:
postulado (1) → ação S deve ser invariante sob transf. Lorentz pois $\delta S = 0$ →

↳ eqs. de movimento.

↳ é interessante escrever S em termos tempo próprio $d\tau$!

como Eq. (184.2) : $dt = \gamma d\tau$, temos que

$$S = \int_{\tau_1}^{\tau_2} d\tau \, \gamma L \tag{224.1}$$

como $d\tau$ invariante → γL invariante (escalar) sob transf. Lorentz !

inicial : considerar partícula livre,

Eq. (223.2) : $\dfrac{dP^\alpha}{d\tau} = \dfrac{d}{d\tau}\left(mU^\alpha\right) = 0$

ou (224.2)

$$\frac{d}{dt}\left(\gamma m\vec{u}\right) = 0 \quad e \quad \frac{dE}{dt} = 0$$

notar :

L depender : m e $\vec{u}$

" NOT " : $\vec{r}$, pois sistema apresenta invariância translacional !

Lembrar invariante de Lorentz relacionado velocidade,

Eq. (192.1) : $U_\alpha U^\alpha = c^2$

↳ $\gamma L \sim U_\alpha U^\alpha = c^2$! De fato, temos que :

$$\gamma L = -mc^2 \quad ou \quad L_{free} = -\frac{mc^2}{\gamma} = -mc^2\left(1 - \frac{u^2}{c^2}\right)^{1/2} : \tag{224.3}$$

: Lagrangiana relativística partícula livre !

notar: $\frac{\partial L}{\partial r_i} = 0$ e $\frac{\partial L}{\partial \dot{r}_i} = \frac{\partial L}{\partial u_i} = -\gamma mc^2 \cdot 1/2 \left(\frac{-2u_i}{c^2}\right) = \gamma m u_i$

↳ Eq. de Lagrange (221.4) : $\frac{d}{dt}(\gamma m \vec{u}) = 0$: ok c/ (224.2)

como Eq. (193.3) : $\mathcal{E} = \left((pc)^2 + (mc^2)^2\right)^{1/2}$

↳ $\frac{d\mathcal{E}}{dt} = \frac{1}{\mathcal{E}} \cdot 1/2 \cdot 2pc^2 \frac{dp}{dt} = 0$, pois $\dot{p} = 0$: ok c/ (224.2) (225.1)

p/ $u/c \ll 1$, temos que

$$L_{free} \simeq -mc^2\left(1 - \frac{1}{2}\frac{u^2}{c^2} - \frac{1}{8}\frac{u^4}{c^4} + \dots\right) = -mc^2 + \frac{1}{2}mu^2 + \frac{1}{8}m\frac{u^4}{c^2} + \dots \quad (225.2)$$

· considerar: partícula sob campos $\vec{E}$ e $\vec{B}$,

Eq. (223.1) → $\gamma L \sim$ produto escalar 4-vetor $A^\alpha = (\Phi, \vec{A})$ e
4-velocidade $U^\alpha = (\gamma c, \gamma \vec{u})$

novamente: invariância translacional exclui $x^\alpha = (x^0, \vec{r})$!

De fato, temos que

$$\gamma L = -mc^2 - \frac{1}{c} q\, U^\alpha A_\alpha$$

↳ $L = -mc^2\left(1 - \frac{u^2}{c^2}\right)^{1/2} - q\Phi + q\frac{\vec{u}}{c}\cdot\vec{A}$: (225.3)

: Lagrangiana relativística, partícula
sob campo EM

notar: (i) p/ $\frac{u}{c} \ll 1$: Eqs. (225.2) e (225.3) $\longrightarrow$ (223.1) !

(ii) $$\frac{\partial L}{\partial x_i} = -q\,\partial_i \Phi + q\,\frac{1}{c}\,u_j \partial_i A_j$$

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial u_i} = -\gamma m c^2 \cdot 1/2 \cdot \frac{(-2u_i)}{c^2} + \frac{1}{c}\,q A_i$$

Eq. (221.4) : $$-q\,\partial_i \Phi + \frac{1}{c}\,q\,u_j \partial_i A_j - \frac{d}{dt}\left(\gamma m u_i + \frac{1}{c}\,q A_i\right) = 0$$

como $\frac{d}{dt}(\gamma m u_i) = \frac{dp_i}{dt}$

e $\frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + \vec{u}\cdot\vec{\nabla} A_i = \partial_t A_i + u_j \partial_j A_i$

↳ $$\frac{dp_i}{dt} = q\left(-\partial_i \Phi - \frac{1}{c}\,\partial_t A_i\right) + \frac{1}{c}\,q\left(u_j \partial_i A_j - u_j \partial_j A_i\right)$$

como $\left(\vec{u}\times(\vec{\nabla}\times\vec{A})\right)_i = \epsilon_{ijk}\,u_j\,(\vec{\nabla}\times\vec{A})_k = \epsilon_{ijk}\,\epsilon_{k\alpha\beta}\,u_j\,\partial_\alpha A_\beta$

$= u_j \partial_i A_j - u_j \partial_j A_i$

pg. 39.1

↳ $$\frac{dp_i}{dt} = q\left(E_i + \frac{1}{c}(\vec{u}\times\vec{B})_i\right)$$: OK c/ Eq. (223.2) !

(iii) Eq. (225.1) : $$\frac{d\varepsilon}{dt} = \frac{c^2}{\varepsilon}\,\vec{p}\cdot\frac{d\vec{p}}{dt} = \frac{qc^2}{\varepsilon}\,\vec{p}\cdot\left(\vec{E} + \frac{1}{c}\,\vec{u}\times\vec{B}\right)$$

↳ $$\frac{d\varepsilon}{dt} = q\,\frac{c^2}{\varepsilon}\,\vec{p}\cdot\vec{E} = q\,\underbrace{\frac{\gamma m c^2}{\varepsilon}}_{1}\,\vec{u}\cdot\vec{E}$$: OK c/ Eq. (223.2) !

· Eqs. (222.2) e (225.3) :

$$\pi_i = \frac{\partial L}{\partial u_i} = \gamma m u_i + \frac{1}{c} q A_i = p_i + \frac{1}{c} q A_i \neq p_i = \gamma m u_i$$

↳ $\vec{\pi} = \vec{p} + \frac{1}{c} q \vec{A}$: momento canonicamente conjugado (227.1)

Eq. (222.3) :

$$H = \pi_i u_i - L = \vec{u} \cdot \vec{\pi} - L$$

$$= \vec{u} \cdot \left(\gamma m \vec{u} + \cancel{\frac{1}{c} q \vec{A}} \right) + \frac{mc^2}{\gamma} + q\Phi - \cancel{q \frac{\vec{u}}{c} \cdot \vec{A}}$$

$$= \frac{mc^2}{\gamma} \Big(\underbrace{1 + \frac{\gamma^2 u^2}{c^2}}_{= \gamma^2} \Big) + q\Phi = mc^2 \gamma + q\Phi = \mathcal{E} + q\Phi$$

Eq. (193.3)

$$= \left((pc)^2 + m^2 c^4 \right)^{1/2} + q\Phi$$

↳ $H = \left((c\vec{\pi} - q\vec{A})^2 + m^2 c^4 \right)^{1/2} + q\Phi$: Hamiltoniana relativística partícula sob campo EM (227.2)

notar : como $H = \mathcal{E} + q\Phi$ → H : energia <u>total</u> partícula!

Exercício : verificar que Eqs. (222.4) e (227.2) ⇒ Eq. (223.2)

notar : $(H - q\Phi)^2 = (c\vec{\pi} - q\vec{A})^2 + m^2 c^4$

$$= c^2 \left(\vec{\pi} - q\vec{A}/c \right)^2 + m^2 c^4$$

comparando c/ Eq. (193.3), podemos definir 4-vetor

$$P^{\alpha} = \left(\frac{1}{c}(H - q\Phi), \vec{\pi} - \frac{1}{c}q\vec{A}\right)$$

$$= \left(\frac{H}{c}, \vec{\pi}\right) - \frac{1}{c}q(\Phi, \vec{A})$$

$$\equiv \pi^{\alpha} - \frac{1}{c}q A^{\alpha}$$: 4-momento canonicamente conjugado (228.1)

: comparar c/ Eq. (227.1)

Obs.: veja comentário pg. 228.1 !

· Movimento partícula carregada sob campos EM estáticos,

vimos que, eqs. de movimento partícula massa m e carga q sob campo EM, Eq. (223.2):

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right) \quad ; \quad \vec{p} = \gamma m\vec{v}$$

$$\frac{d\mathcal{E}}{dt} = q\,\vec{v}\cdot\vec{E} \quad ; \quad \mathcal{E} = \gamma mc^2 \qquad (228.2)$$

vamos considerar 4 casos particulares,

(1) $\vec{E} = cte$ e $\vec{B} = 0$

hipótese $\vec{E} = E\hat{x}$

↳ eqs. de movimento (228.2) :

$$\frac{d\vec{p}}{dt} = qE\hat{x} \qquad (228.3)$$

Obs.: o procedimento utilizado acima (descrito nas Sec. 12.1.A, Jackson e §16 Landau) para determinar a Lagrangiana (225.3) não é único;
nesse caso, a ideia foi preservar a eq. de Lagrange (221.4);

p/ procedimentos alternativos, que consideram o vínculo $U_\mu U^\mu = c^2$, veja Sec. 12.1.B, Jackson e Sec. 2.2, Barut

↳ $\dot{p}_x = \frac{d}{dt}(\gamma m v_x) = qE$

(229.1)

$\dot{p}_y = \frac{d}{dt}(\gamma m v_y) = 0$ e $\dot{p}_z = \frac{d}{dt}(\gamma m v_z) = 0$

notar Eqs. (229.1) : movimento partícula ∈ plano

↳ hipótese : condição inicial $\vec{v}_0 = v_{0x}\hat{x} + v_{0y}\hat{y}$

↳ movimento ∈ plano xy !

temos que : $\gamma m v_x = qEt + c_1$

$\gamma m v_y = c_2$

$\gamma m v_z = c_3$

c.i. : $\gamma_0 m v_{0x} = c_1$; $\gamma_0 m v_{0y} = c_2$; $c_3 = 0$

onde $\gamma_0 = \left(1 - \frac{v_0^2}{c^2}\right)^{-1/2}$

↳ $\gamma v_x = \frac{qE}{m}t + \gamma_0 v_{0x} \equiv at + \gamma_0 v_{0x}$

(229.2)

$\gamma v_y = \gamma_0 v_{0y}$ e $v_z = 0$

próxima etapa : determinar γ,

notar :

$\gamma^2(v_x^2 + v_y^2) = a^2t^2 + 2a\gamma_0 v_{0x} t + \gamma_0^2(v_{0x}^2 + v_{0y}^2)$

$\div 1/c^2$: $\gamma^2\beta^2 = \gamma^2 - 1 = \frac{a^2t^2}{c^2} + \frac{2a\gamma_0 v_{0x} t}{c^2} + \gamma_0^2 - 1$

↳ $\gamma = \frac{1}{c}\left((at)^2 + 2at\gamma_0 v_{0x} + c^2\gamma_0^2\right)^{1/2}$ (229.3)

Eqs. (229.2) e (229.3) :

$$\frac{v_x}{c} = \frac{at + \gamma_0 v_{0x}}{(a^2t^2 + 2at\gamma_0 v_{0x} + c^2\gamma_0^2)^{1/2}}$$

(230.1)

$$\frac{v_y}{c} = \frac{\gamma_0 v_{0y}}{(a^2t^2 + 2at\gamma_0 v_{0x} + c^2\gamma_0^2)^{1/2}}$$

como $v_x = \frac{dx}{dt}$ e $v_y = \frac{dy}{dt}$, verifica-se que (exercicio):

$$x(t) = \frac{c}{a}\left(a^2t^2 + 2at\gamma_0 v_{0x} + c^2\gamma_0^2\right)^{1/2} + C_0$$

(230.2)

$$y(t) = \frac{c}{a}\gamma_0 v_{0y} \sinh^{-1}\left(\frac{at + \gamma_0 v_{0x}}{\gamma_0 (c^2 - v_{0x}^2)^{1/2}}\right) + C_1 \quad ; \quad a = qE/m$$

onde ctes C_0 e C_1 ~ c.i.

Eq. (230.2) : eq. paramétrica trajetória partícula.

definindo : $\bar{x} = x - C_0$ e $\bar{y} = y - C_1$, verifica-se que (230.2) pode ser escrita como (exercício):

$$\bar{x} = \frac{c\gamma_0 m}{qE}(c^2 - v_{0x}^2)^{1/2} \cosh\left(\frac{qE\bar{y}}{mc\gamma_0 v_{0y}}\right) \quad : \text{Eq. catenária !} \qquad (230.3)$$

notar:

(i) Eq. (230.1) : $t \to +\infty \Rightarrow v_x \to c$ e $v_y \to 0$;

(ii) Eq. (230.3), limite $v \ll c$:

$$\bar{x} \simeq \frac{c^2 m}{qE}\cosh\left(\frac{qE\bar{y}}{mcv_{0y}}\right) \simeq \frac{c^2 m}{qE}\left(1 + \frac{1}{2}\left(\frac{qE\bar{y}}{mcv_{0y}}\right)^2\right)$$

↳ $\bar{x} \simeq \frac{qE}{mv_{0y}^2}\, y^2 + cte$: parábola !

(2) $\vec{E} = 0$ e $\vec{B} = cte$

Eqs. de movimento (228.2) : $\frac{d\vec{p}}{dt} = \frac{1}{c} q \vec{v} \times \vec{B}$

$\frac{d\mathcal{E}}{dt} = \frac{d}{dt}(\gamma m c^2) = 0 \rightarrow v = |\vec{v}| = cte$!

temos que:

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m \vec{v}) = \gamma m \frac{d\vec{v}}{dt} = \frac{1}{c} q \vec{v} \times \vec{B}$$

↳ $\frac{d\vec{v}}{dt} = \vec{v} \times \left(\frac{q\vec{B}}{\gamma m c}\right) \equiv \vec{v} \times \vec{\omega}_B$: similar problema não-relativístico, porém frequência ciclotrônica $\omega_c = qB/mc \rightarrow \omega_B$! (231.1)

notar : $\vec{\omega}_B = \frac{q\vec{B}}{\gamma m c} = \frac{qc}{\mathcal{E}} \vec{B}$

hipótese : $\vec{B} = B\hat{z}$ e c.i. : $\vec{r}(t=0) = (x_0, y_0, z_0)$

$\vec{v}(t=0) = (v_{0x}, v_{0y}, v_{0z})$

Eq. (231.1) :
$$\left.\begin{aligned} \dot{v}_x &= \omega_B v_y \\ \dot{v}_y &= -\omega_B v_x \end{aligned}\right\}$$ 2 eq. diferenciais acopladas

$$\dot{v}_z = 0$$

· direção $\hat{z}$:

$\dot{v}_z = 0 \rightarrow v_z = C_0 = \frac{dz}{dt} \rightarrow z = C_0 t + C_1$

c.i. : $z(0) = z_0 \rightarrow C_0 = v_{0z} \rightarrow z(t) = v_{0z} t + z_0$:

$v_z(0) = v_{0z} \quad C_1 = z_0$ movimento uniforme!

(232.1)

direções $\hat{x}$ e $\hat{y}$

$\hookrightarrow \dot{v}_x + i\dot{v}_y = -i\omega_B (v_x + i v_y)$

$\hookrightarrow v_x + i v_y = A e^{-i(\omega_B t + \delta)}$; $A \in \mathbb{R}$

ou $v_x(t) = A\cos(\omega_B t + \delta)$

$v_y(t) = -A\sin(\omega_B t + \delta)$

c.i. : $v_x(0) = v_{0x} = A\cos\delta$

$v_y(0) = v_{0y} = -A\sin\delta$

como : $v_x = A\cos\delta\cos\omega_B t - A\sin\delta\sin\omega_B t$

$v_y = -A\cos\delta\sin\omega_B t - A\sin\delta\cos\omega_B t$

$\hookrightarrow v_x(t) = v_{0x}\cos\omega_B t + v_{0y}\sin\omega_B t$ (232.2)

$v_y(t) = -v_{0x}\sin\omega_B t + v_{0y}\cos\omega_B t$

como :

$v_x = \frac{dx}{dt} \rightarrow x = \frac{v_{0x}}{\omega_B}\sin\omega_B t - \frac{v_{0y}}{\omega_B}\cos\omega_B t + C_1$

$v_y = \frac{dy}{dt} \rightarrow y = \frac{v_{0x}}{\omega_B}\cos\omega_B t + \frac{v_{0y}}{\omega_B}\sin\omega_B t + C_2$

c.i. : $x(0) = x_0 = -v_{0y}/\omega_B + C_1 \rightarrow C_1 = x_0 + v_{0y}/\omega_B$

$y(0) = y_0 = v_{0x}/\omega_B + C_2 \rightarrow C_2 = y_0 - v_{0x}/\omega_B$

Dessa forma.

$$x(t) - \left(x_0 + v_{0y}/\omega_B\right) = \frac{v_{0x}}{\omega_B}\sin\omega_B t - \frac{v_{0y}}{\omega_B}\cos\omega_B t$$

$$y(t) - \left(y_0 - v_{0x}/\omega_B\right) = \frac{v_{0x}}{\omega}\cos\omega_B t + \frac{v_{0y}}{\omega_B}\sin\omega_B t \qquad (233.1)$$

$z(t) - z_0 = v_{0z}t$: equação paramétrica hélice.

eixo = $\hat{z}$: direção $\vec{B}$!

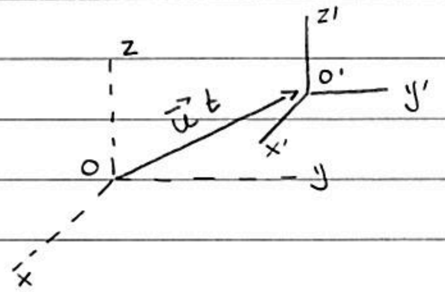
notar : movimento plano xy : movimento circular.

$$\text{raio } a = \frac{1}{\omega_B}\left(v_{0x}^2 + v_{0y}^2\right)^{1/2} = \frac{v_{0\perp}}{\omega_B} = \frac{\gamma m c v_{0\perp}}{qB} = \frac{c p_{0\perp}}{qB}.$$

(3) $\vec{E} = \text{cte}$, $\vec{B} = \text{cte}$ e $\vec{E} \perp \vec{B}$.

ideia: analisar o movimento da partícula REF K', escolhido de modo a simplificar eqs. de movimento (228.2)

considerar REFs K e K'.



$\vec{u}$: velocidade REF K' w.r.t. REF K

eixos $\hat{e}_i \parallel$ eixos $\hat{e}'_i$

eqs. de movimento (228.2):

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right) \quad ; \quad \vec{E} = \text{cte},\ \vec{B} = \text{cte},\ \vec{E}\cdot\vec{B} = 0 \ : \text{REF } K$$

$$\frac{d\vec{p}\,'}{dt'} = q\left(\vec{E}' + \frac{1}{c}\vec{v}\,'\times\vec{B}'\right) \quad : \text{REF } K'$$

Lembrar P11.14, Jackson:

$$\vec{E}\cdot\vec{B} = -1/4\, F^{\alpha\beta}\tilde{F}_{\alpha\beta} \qquad : \text{escalares, invariantes} \qquad (234.1)$$
$$E^2 - B^2 = -1/2\, F^{\alpha\beta}F_{\alpha\beta} \qquad \text{sob transf. de Lorentz}$$

↳ $E^2 - B^2 = E'^2 - B'^2$ e $\vec{E}\cdot\vec{B} = \vec{E}'\cdot\vec{B}'$

notar: (i) se $E' = 0$

↳ $\vec{E}\cdot\vec{B} = \vec{E}'\cdot\vec{B}' = 0$: OK

e $E^2 - B^2 = -B'^2 \rightarrow E^2 - B^2 < 0$ ou $E < B$

↳ se $E < B \rightarrow$ é possível escolher REF K' tal que $\vec{E}' = 0$

(ii) se $B' = 0$

↳ $\vec{E}\cdot\vec{B} = \vec{E}'\cdot\vec{B}' = 0$: OK

e $E^2 - B^2 = E'^2 \rightarrow E^2 - B^2 > 0$ ou $E > B$

↳ se $E > B \rightarrow$ é possível escolher REF K' tal que $\vec{B}' = 0$

vamos analisar os 2 casos separadamente,

(i) hipótese $E < B$,

ideia: determinar velocidade $\vec{u}$ REF K' tal que $\vec{E}' = 0$

Lembrar: transf. campos $\vec{E}$ e $\vec{B}$, Eq. (217.2):

$$\vec{E}' = \gamma(\vec{E} + \vec{\beta}\times\vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E}) \qquad \vec{\beta} = \vec{u}/c$$

$$\vec{B}' = \gamma(\vec{B} - \vec{\beta}\times\vec{E}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B}) \qquad \gamma = (1 - u^2/c^2)^{-1/2}$$

$\vec{E}' = 0 \rightarrow (\gamma+1)(\vec{E} + \vec{\beta}\times\vec{B}) = \gamma\vec{\beta}(\vec{\beta}\cdot\vec{E})$

se $\vec{u} \perp \vec{E} \rightarrow \vec{E} + \vec{\beta}\times\vec{B} = 0 \rightarrow \vec{B}\times\vec{E} = -\vec{B}\times(\vec{\beta}\times\vec{B}) = \vec{B}(\vec{\beta}\cdot\vec{B}) - B^2\vec{\beta}$

↳ se $\vec{u} \perp \vec{B} \rightarrow \vec{u} = c\dfrac{1}{B^2}\vec{E}\times\vec{B}$ (235.1)

Eq. (235.1) : possível escolha p/ $\vec{u}$ tal que $\vec{E}' = 0$

notar : $\vec{B}' = \gamma(\vec{B} - \vec{\beta}\times\vec{E}) = \gamma\left(\vec{B} - \underbrace{\dfrac{1}{B^2}(\vec{E}\times\vec{B})\times\vec{E}}\right)$

$+1/B^2\,\vec{E}\times(\vec{E}\times\vec{B}) = 1/B^2\left(\underbrace{(\vec{E}\cdot\vec{B})}_{=0}\vec{E} - E^2\vec{B}\right)$

↳ $\vec{B}' = \gamma\left(1 - \dfrac{E^2}{B^2}\right)\vec{B}$

como $\beta^2 = \dfrac{1}{B^4}(\vec{E}\times\vec{B})\cdot(\vec{E}\times\vec{B}) = \dfrac{1}{B^4}E^2\cdot B^2 = \dfrac{E^2}{B^2} \rightarrow \gamma = \left(1 - \dfrac{E^2}{B^2}\right)^{-1/2}$

↳ $\vec{B}' = \left(1 - \dfrac{E^2}{B^2}\right)^{1/2}\vec{B} = \dfrac{1}{\gamma}\vec{B}$ (235.2)

notar : $\vec{B}' \parallel \vec{B}$ e $B' < B$!

↳ Eq. de movimento REF K' : $\dfrac{d\vec{p}'}{dt'} = q\dfrac{1}{c}\vec{v}'\times\vec{B}'$ = eq. de movimento (231.1) : caso (2) !

temos que :

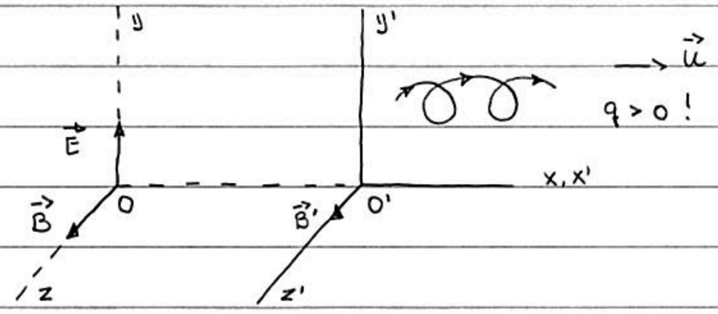
trajetória partícula REF K' : hélice , eixo simetria = direção $\vec{B}'$

" " REF K : " " " $\vec{B}$

⊕ deslocamento lateral

direção $\vec{u}$: ⊥ plano $\vec{E}$ - $\vec{B}$

hipótese : $\vec{E} = E\hat{y}$, $\vec{B} = B\hat{z}$ $\rightarrow$ $\vec{u} = u\hat{x}$



(ii) hipótese $E > B$,

ideia : determinar velocidade $\vec{u}$ REF K' tal que $\vec{B}' = 0$

de modo análogo ao caso anterior, verifica-se que uma possivel escolha é

$$\vec{u} = c \frac{1}{E^2} \vec{E} \times \vec{B} \qquad (236.1)$$

notar : $\vec{u} \perp \vec{E}$; $\vec{u} \perp \vec{B}$ e

$$\vec{B}' = \gamma\left(\vec{B} + \frac{1}{E^2} \underbrace{\vec{E} \times (\vec{E} \times \vec{B})}_{-E^2\vec{B}} \right) = 0$$

$$\vec{E}' = \gamma\left(\vec{E} - \frac{1}{E^2} \underbrace{\vec{B} \times (\vec{E} \times \vec{B})}_{B^2\vec{E}} \right) = \gamma\left(1 - \frac{B^2}{E^2}\right)\vec{E} =$$

$$= \left(1 - \frac{B^2}{E^2}\right)^{1/2} \vec{E} = \frac{1}{\gamma}\vec{E} \qquad (236.2)$$

notar $\vec{E}' \parallel \vec{E}$ e $E' < E$!

↳ Eq. de movimento REF K' : $\frac{d\vec{p}'}{dt'} = q\vec{E}'$ = eq. de movimento (228.3) = caso (1) !

(veja P 12.3, Jackson !)

Obs.: veja Sec. 22, Landau p/ discussão sobre o caso
$E = B$ e $\vec{E}\cdot\vec{B} = 0$.

(4) $\vec{E} = cte$, $\vec{B} = cte$ e $\vec{E}\cdot\vec{B} \neq 0$.

Eq. (234.1) : $\vec{E}\cdot\vec{B} = \vec{E}'\cdot\vec{B}' \rightarrow EB\cos\theta = E'B'\cos\theta'$

notas: se $0 \leq \theta < \pi/2 \rightarrow 0 \leq \theta' < \pi/2$: sinal $\cos\theta$ é invariante sob transf. Lorentz !
se $\pi/2 < \theta \leq \pi \rightarrow \pi/2 < \theta' \leq \pi$

vamos verificar que:

- se $0 \leq \theta < \pi/2 \rightarrow$ é possível determinar REF K' tal que $\theta' = 0$ ou $\vec{E}' = \alpha\vec{B}'$, $\alpha > 0$
- se $\pi/2 < \theta \leq \pi \rightarrow$ " " " " " $\theta' = \pi$ ou $\vec{E}' = -\alpha\vec{B}'$, $\alpha > 0$

hipótese (i): $\vec{E}'\times\vec{B}' = 0$ e $\vec{u}\cdot\vec{E} = \vec{u}\cdot\vec{B} = 0$

↳ transf. campos $\vec{E}$ e $\vec{B}$ (217.2):

$$\vec{E}' = \gamma(\vec{E} + \vec{\beta}\times\vec{B}) \quad \text{e} \quad \vec{B}' = \gamma(\vec{B} - \vec{\beta}\times\vec{E})$$

Verifica-se que (exercício):

$$\vec{E}'\times\vec{B}' = \gamma^2(\vec{E} + \vec{\beta}\times\vec{B})\times(\vec{B} - \vec{\beta}\times\vec{E})$$

escolha!

$$= \gamma^2\left(\vec{E}\times\vec{B} - (E^2 + B^2)\vec{\beta} + \vec{\beta}\left(\vec{\beta}\cdot(\vec{E}\times\vec{B})\right)\right) = 0 \qquad (237.1)$$

hipótese (ii) : $\vec{\beta} = \frac{\vec{u}}{c} = \lambda \frac{\vec{E}\times\vec{B}}{EB}$, onde $\lambda = \text{cte}$ (238.1)

Eqs. (237.1) e (238.1) :

$$\gamma^2(\vec{E}\times\vec{B})\left(1 - \frac{(E^2+B^2)}{EB}\lambda + \underbrace{\frac{\lambda^2}{(EB)^2}\underbrace{(\vec{E}\times\vec{B})\cdot(\vec{E}\times\vec{B})}_{E^2B^2-(\vec{E}\cdot\vec{B})^2}}_{\lambda^2\sin^2\theta}\right) = 0$$

↳ $\lambda^2 EB\sin^2\theta - (E^2+B^2)\lambda + EB = 0$: eq. 2º grau p/ λ !

$\exists$ sol. $\lambda \in \mathbb{R} \to (E^2+B^2)^2 - 4(EB)^2\sin^2\theta = \Delta > 0$

De fato : $\Delta = E^4 + B^4 + 2E^2B^2(\underbrace{1 - 2\sin^2\theta}_{=-1+2\cos^2\theta\,=\,\cos 2\theta})$

$$> E^4 + B^4 - 2E^2B^2 = (E^2-B^2)^2 > 0 \;! \qquad (238.2)$$

Eq. (238.2) : indica que é possivel escolher REF K'
tal que velocidade $\vec{u}$ REF K' w.r.t. REF K =
= Eq. (238.1);
nesse caso, REF K' é tal que $\vec{E}' \parallel \vec{B}'$!

Obs. : p/ trajetória partícula, veja P 12.6, Jackson.

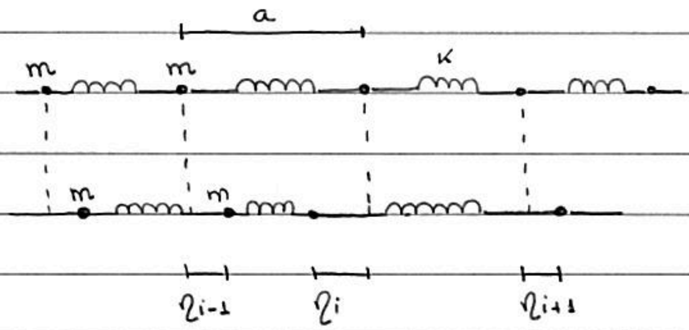
↳

· Lagrangiana do campo EM.

ideia: determinar Lagrangiana do sistema campo EM ⊕ fontes

inicial: revisão formalismo Lagrangiano para meios contínuos
(p/ detalhes, veja Sec. 13.2, Goldstein, 3ª ed.)

considerar: sistema 1-D, N partículas massas m conectadas por molas, cte mola k; distância entre partículas equilíbrio $= a$.



: configuração equilíbrio

: " força "

se $\eta_i = \eta_i(t)$: deslocamento partícula i w.r.t. posição equilíbrio

↳ $L = T - U = \frac{1}{2} \sum_{i=1}^{N} m \dot{\eta}_i^2 - k(\eta_{i+1} - \eta_i)^2$: Lagrangiana sistema discreto (239.1)

Eq. (239.1) pode ser escrita como

$$L = \frac{1}{2} \sum_{i=1}^{N} a \left(\frac{m}{a} \dot{\eta}_i^2 - ka \frac{1}{a^2} (\eta_{i+1} - \eta_i)^2 \right) = \sum_{i=1}^{N} a L_i$$

considerando limite $N \to +\infty$, $a \to 0$, verifica-se que

$L = \sum_{i=1}^{N} a L_i \longrightarrow \int dx \, \mathcal{L}$: Lagrangiana sistema contínuo

onde (239.2)

$\mathcal{L} = \frac{1}{2} \mu \left(\frac{\partial \eta}{\partial t} \right)^2 - \frac{1}{2} Y \left(\frac{\partial \eta}{\partial x} \right)^2$: densidade de Lagrangiana

nesse caso : $\eta_i(t) \rightarrow \eta(x,t)$
$m/a \rightarrow \mu$: densidade de massa
Y : módulo de Young.

notar : $\mathcal{L} = \mathcal{L}(\partial_x \eta, \partial_t \eta)$!

caso geral : considerar sistema 1-D (discreto) descrito pelas coordenadas generalizadas $q_1, q_2, \dots, q_N$;
a generalização p/ caso $N \rightarrow +\infty$ pode ser obtida via as identificações :

$i \rightarrow x$ ou ($ai \rightarrow x$)
$q_i(t) \rightarrow \phi(x,t)$: campo escalar
$\dot{q}_i(t) \rightarrow \partial_t \phi(x,t)$ (240.1)

se $L = \sum_{i=1}^{N} L_i \rightarrow \int dx\, \mathcal{L}$

onde (caso geral) $\mathcal{L} = \mathcal{L}(\phi, \partial_x\phi, \partial_t\phi, x, t)$: densidade de Lagrangiana

Além disso, a ação (221.1) assume a forma

$$S = \int_{-\infty}^{+\infty} dx \int_1^2 dt\, \mathcal{L} \qquad (240.2)$$

e o <u>princípio de Hamilton</u> p/ meios contínuos é dado por

$$\delta S = \delta \int dx\,dt\, \mathcal{L} = 0 \qquad (240.3)$$

próxima etapa: determinar eq. de Lagrange p/ campo escalar $\phi = \phi(x,t)$ a partir (240.3)

ideia: seguir procedimento análogo pg. 221.1.

seja $\phi_0(x,t)$: configuração campo escalar que corresponde extremo ação S;

considerar variação: $\phi(x,t) = \phi_0(x,t) + \alpha\, \xi(x,t)$

tal que $\xi(x \to \pm\infty, t) = \xi(x, t = t_1) = \xi(x, t = t_2) = 0$

↳ $\mathcal{L} = \mathcal{L}(\phi_0 + \alpha\xi,\ \partial_x\phi_0 + \alpha\partial_x\xi,\ \partial_t\phi_0 + \alpha\partial_t\xi,\ x,\ t)$

↳ $S = S[\alpha]$

nesse caso, δS pode ser determinado via

$$\frac{dS}{d\alpha} = \int dx\,dt \left(\frac{\partial\mathcal{L}}{\partial\phi} \underbrace{\frac{\partial\phi}{\partial\alpha}}_{\xi} + \underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_x\phi)} \underbrace{\frac{\partial}{\partial\alpha}(\partial_x\phi)}_{\partial_x\xi}}_{-\partial_x\left(\frac{\partial\mathcal{L}}{\partial(\partial_x\phi)}\right)\xi + \underbrace{\frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial(\partial_x\phi)}\xi\right)}_{=0,\ \text{pois } \xi(x\to\pm\infty,t)=0}} + \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} \underbrace{\frac{\partial}{\partial\alpha}(\partial_t\phi)}_{\partial_t\xi} \right)$$

procedendo de modo análogo p/ o terceiro termo, temos que

$$\frac{dS}{d\alpha} = \int dx\,dt \left(\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial(\partial_x\phi)}\right) - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\right) \right) \xi(x,t)$$

como $\xi = \xi(x,t)$ é arbitrário, a condição $dS/d\alpha = 0$

↳ $$\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial x)}\right) - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial t)}\right) = 0 \quad : \qquad (241.1)$$

: Eq. de Lagrange

como $\frac{\partial}{\partial x}$ e $\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial x^\mu}$ e Eq. (199.4) : $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$

↳ Eq. (241.1) : $$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) = 0 \qquad (242.1)$$

notar: Eqs. (240.2) e (242.1) podem ser generalizadas p/ caso sistema descrito campos $\phi_i(\vec{r}, t)$; $i = 1, 2, \ldots n$:

$$S = \int d^3r\, dt\, \mathcal{L} = \int d^4x\, \mathcal{L}$$

onde $\mathcal{L} = \mathcal{L}(\phi_i, \partial_\mu \phi_i, x^\alpha, t)$

e $$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \right) = 0 \quad ; \ i = 1, 2, \ldots, n \quad \mu = 0, 1, 2, 3 \qquad (242.2)$$

· caso particular: campo EM.

nesse caso: $\phi_i(\vec{r}, t) \rightarrow A^\alpha(\vec{r}, t)$

$\partial_\mu \phi_i \rightarrow \partial_\mu A^\alpha$

Lembrar: $$d^4x' = \frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)} d^4x = \underbrace{(\det A)}_{=1 \text{ p/ transf. Lorentz próprias (veja pg. 202)}} d^4x = d^4x$$

como d^4x é invariante sob transf. Lorentz
↳ ação S " " " " se $\mathcal{L}$ é um <u>escalar</u>!

vamos considerar 2 casos a fim de determinar a densidade de Lagrangiana,

(i) ausência fontes externas.

nesse caso, a analogia c/ sistema partículas, e.g., Eq. (239.2)

↳ $\mathcal{L} \sim$ termo de energia cinética $\sim (\partial_\mu \phi_i)^2 \rightarrow (\partial_\mu A^\alpha)^2$

Lembrar: Eq. (211.2) : $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$

e Eqs. (234.1) : $F^{\alpha\beta} \tilde{F}_{\alpha\beta} = -4 \vec{E} \cdot \vec{B}$: invariantes
$F^{\alpha\beta} F_{\alpha\beta} = -2(E^2 - B^2)$ sob transf. Lorentz

entretanto, sob inversão espacial (paridade), temos que

Eq. (74.2) : $\vec{E} \rightarrow -\vec{E}$ e $\vec{B} \rightarrow +\vec{B}$

↳ sob inversão espacial : $F^{\alpha\beta} \tilde{F}_{\alpha\beta} \rightarrow -F^{\alpha\beta} \tilde{F}_{\alpha\beta}$: pseudoescalar
$F^{\alpha\beta} F_{\alpha\beta} \rightarrow +F^{\alpha\beta} F_{\alpha\beta}$: escalar

↳ $\mathcal{L} \propto F^{\alpha\beta} F_{\alpha\beta}$: escalar sob transf. Lorentz e
" inversão espacial

verifica-se que (veja abaixo):

$$\mathcal{L} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} \tag{243.3}$$

(ii) ⊕ fontes externas.

nesse caso, $\mathcal{L} \sim J^\alpha = (c\rho, \vec{J})$

vimos que, p/ partícula carregada sob campo EM,
$L \sim U^\alpha A_\alpha$: Eq. (225.3);

novamente, analogia c/ sistema partículas

↳ $\mathcal{L} \propto J_\alpha A^\alpha$

verifica-se que (veja abaixo) : $\mathcal{L} = -\frac{1}{c} J_\alpha A^\alpha$ (244.1)

↳ Eqs. (243.3) e (244.1) :

$$\mathcal{L} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha$$: densidade de Lagrangiana campo EM (244.2)

próxima etapa: determinar eqs. de movimento p/ A^α,

como: $F_{\mu\nu} F^{\mu\nu} = g_{\mu\lambda} g_{\nu\sigma} F^{\lambda\sigma} F^{\mu\nu}$

$$= g_{\mu\lambda} g_{\nu\sigma} (\partial^\lambda A^\sigma - \partial^\sigma A^\lambda)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

↳ $$\mathcal{L} = -\frac{1}{16\pi} g_{\mu\lambda} g_{\nu\sigma} (\partial^\lambda A^\sigma - \partial^\sigma A^\lambda)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{c} J_\mu A^\mu$$

Eq. de Lagrange (242.2) : $$\frac{\partial \mathcal{L}}{\partial A^\alpha} - \partial^\beta \left(\frac{\partial \mathcal{L}}{\partial (\partial^\beta A^\alpha)} \right) = 0$$ (244.3)

como: $$\frac{\partial \mathcal{L}}{\partial A^\alpha} = -\frac{1}{c} J_\alpha$$

e $$\frac{\partial \mathcal{L}}{\partial (\partial^\beta A^\alpha)} = -\frac{1}{16\pi} g_{\mu\lambda} g_{\nu\sigma} \left(\delta_\beta{}^\lambda \delta_\alpha{}^\sigma F^{\mu\nu} - \delta_\beta{}^\sigma \delta_\alpha{}^\lambda F^{\mu\nu} + \right.$$

$$\left. + \delta_\beta{}^\mu \delta_\alpha{}^\nu F^{\lambda\sigma} - \delta_\beta{}^\nu \delta_\alpha{}^\mu F^{\lambda\sigma} \right)$$

$$= -\frac{1}{16\pi} \left(g_{\mu\beta} g_{\nu\alpha} F^{\mu\nu} - g_{\mu\alpha} g_{\nu\beta} F^{\mu\nu} + g_{\beta\lambda} g_{\alpha\sigma} F^{\lambda\sigma} - g_{\alpha\lambda} g_{\beta\sigma} F^{\lambda\sigma} \right)$$

$$= -\frac{1}{16\pi}\left(F_{\beta\alpha} - F_{\alpha\beta} + F_{\beta\alpha} - F_{\alpha\beta} \right)$$

$$= -\frac{1}{4\pi} F_{\beta\alpha} = \frac{1}{4\pi} F_{\alpha\beta} \qquad (245.1.0)$$

↳ Eq. (244.3) : $\frac{1}{4\pi}\partial^\beta F_{\beta\alpha} = \frac{1}{c} J_\alpha$: Eq. (213.1) : (*)

forma covariante eqs. de Maxwell não-homogêneas!

notar : (*) → coeficientes Lagrangiana (244.2) ok!

· sobre as equações de Maxwell homogêneas : não são derivadas a partir de (244.2) ; lembrar que eqs. homogêneas estão contidas na definição $F^{\alpha\beta}$ (veja pg. 211) e podem ser escritas como (forma covariante)

$\partial_\alpha \tilde{F}^{\alpha\beta} = 0$: Eq. (214.2) ; notar : 4 equações!

ou (veja pg. 245.1) :

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \quad ; \alpha, \beta, \gamma = 0,1,2,3 \qquad (245.1)$$

notar : 4-divergente (*)

↳ $\frac{1}{c}\partial^\alpha J_\alpha = \frac{1}{4\pi}\partial^\alpha\partial^\beta F_{\alpha\beta} = \frac{1}{8\pi}\left(\partial^\alpha\partial^\beta F_{\alpha\beta} + \underbrace{\partial^\alpha\partial^\beta F_{\alpha\beta}}_{\partial^\beta\partial^\alpha F_{\beta\alpha} = -\partial^\alpha\partial^\beta F_{\alpha\beta}} \right) = 0$

↳ $\partial^\alpha J_\alpha = 0$: forma covariante, eq. de continuidade (209.3)!

tilibra

• vamos derivar Eq. (245.1) a partir Eq. (214.2),

$$\partial_\alpha \tilde{F}^{\alpha\beta} = \frac{1}{2}\partial_\alpha\left(\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu}\right) = 0$$

se $\beta = 0$; $\partial_\alpha \tilde{F}^{\alpha 0} = \frac{1}{2}\partial_\alpha\left(\epsilon^{\alpha 0\mu\nu}F_{\mu\nu}\right) =$

$$= \frac{1}{2}\Big(\partial_1 F_{23} - \partial_1 \underbrace{F_{32}}_{=-F_{23}} + \partial_2 F_{31} - \partial_2 \underbrace{F_{13}}_{=-F_{31}} + \partial_3 F_{12} - \partial_3 \underbrace{F_{21}}_{=-F_{12}}\Big)$$

$$= \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0$$

análogo p/ $\beta = 1,2,3$ $\rightarrow$ Eq. (245.1) !

• Obs.: densidade de Lagrangiana (244.2) pode ser escrita como

$$\mathcal{L} = -\frac{1}{16\pi} g_{\alpha\mu} g_{\beta\nu} (\partial^\alpha A^\beta - \partial^\beta A^\alpha)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{c} J_\alpha A^\alpha$$

$$= \frac{1}{8\pi}(E^2 - B^2) - \rho\Phi + \frac{1}{c}\vec{J}\cdot\vec{A}$$

$$= \frac{1}{8\pi}\left(\left(-\vec{\nabla}\Phi - \frac{1}{c}\partial_t\vec{A}\right)^2 - \left(\vec{\nabla}\times\vec{A}\right)^2\right) - \rho\Phi + \frac{1}{c}\vec{J}\cdot\vec{A} \qquad (245.2)$$

Ex.: densidades de Lagrangiana e eqs. de movimento
(veja Sec. 3.3, Barut)

(1) Campo escalar real,

$$\mathcal{L} = \frac{1}{2}\left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - m^2\phi^2\right) \quad ; \quad \phi = \phi(\vec{x},t) \qquad (245.3)$$

eq. de Lagrange (242.2),

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\right) = 0$$

como $\dfrac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$ e $\dfrac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)} = \dfrac{1}{2}\left(g^{\alpha\nu}\partial_\nu\phi + g^{\mu\alpha}\partial_\mu\phi\right)$

$$\hookrightarrow \quad g^{\alpha\nu}\partial_\alpha\partial_\nu\phi + m^2\phi = 0 \rightarrow \left(\partial^\nu\partial_\nu + m^2\right)\phi = 0 \; : \qquad (245.4)$$

: eq. de Klein-Gordon

verifica-se que (exercício)

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} = \frac{1}{c}\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \frac{1}{c}g^{00}\partial_0\phi = \frac{1}{c^2}\partial_t\phi \; :$$

: densidade momento canonicamente conjugado (253.2)

e $\mathcal{H} = \dfrac{1}{2}\left(c^2\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$: densidade de Hamiltoniana

(2) Campo de Dirac,

$$\mathcal{L} = i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\psi + m\bar{\Psi}\psi \qquad (245.5)$$

onde $\psi^t = (\psi_1\ \psi_2\ \psi_3\ \psi_4)$; $\psi_i = \psi_i(\vec{r},t)$; $\bar{\Psi} = \psi^+\gamma^0$;

$$\gamma^0 = \beta = \begin{pmatrix} \hat{1}_{2\times 2} & 0 \\ 0 & \hat{1}_{2\times 2} \end{pmatrix} \qquad \gamma^i = \beta\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

eqs. de Lagrange (242.2),

$$\frac{\partial \mathcal{L}}{\partial \psi_\lambda} - \partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi_\lambda)}\right) = 0 \quad e \quad \frac{\partial \mathcal{L}}{\partial \psi^+_\lambda} - \partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi^+_\lambda)}\right) = 0$$

como $\mathcal{L} = i\psi^+_\alpha \gamma^0_{\alpha\bar{\alpha}}\gamma^{\mu}_{\bar{\alpha}\beta}\partial_\mu\psi_\beta + m\psi^+_\alpha\gamma^0_{\alpha\beta}\psi_\beta$

$$\hookrightarrow \frac{\partial \mathcal{L}}{\partial \psi^+_\lambda} = i\gamma^0_{\lambda\bar{\alpha}}\gamma^{\mu}_{\bar{\alpha}\beta}\partial_\mu\psi_\beta + m\gamma^0_{\lambda\beta}\psi_\beta$$

$$\frac{\partial \mathcal{L}}{\partial \psi_\lambda} = m\psi^+_\alpha\gamma^0_{\alpha\lambda} \quad e \quad \frac{\partial \mathcal{L}}{\partial(\partial_\nu\psi_\lambda)} = i\psi^+_\alpha\gamma^0_{\alpha\bar{\alpha}}\gamma^{\nu}_{\bar{\alpha}\lambda}$$

$$\hookrightarrow i\left(\gamma^0\gamma^\nu\partial_\nu\psi\right)_\lambda + m(\gamma^0\psi)_\lambda = 0 \rightarrow (i\gamma^\nu\partial_\nu + m)\psi = 0$$

$$\hookrightarrow m(\psi^+\gamma_0)_\lambda - i(\partial_\nu\psi^+\gamma^0\gamma^\nu)_\lambda = 0 \rightarrow i\partial_\nu\bar{\Psi}\gamma^\nu - m\bar{\Psi} = 0$$

(245.6)

· tensores de tensão canônico e simétrico,

ideia: determinar as leis de conservação de energia (60.4), momentos linear (63.1) e angular (66.1) no formalismo Lagrangiano!

hipótese: $\mathcal{L} = \mathcal{L}(\phi_i, \partial_\mu \phi_i)$, i.e.,

$\mathcal{L}$ não depende explicitamente da posição e do tempo
↳ sistema invariante sob translações espacial e temporal $\xrightarrow{\text{(veja abaixo)}}$ momento linear e energia total são conservados!

notar:

$$\frac{\partial \mathcal{L}}{\partial x^\nu} = \partial_\nu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_i} \partial_\nu \phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\nu(\partial_\mu \phi_i)$$

notar!

Eq. de Lagrange (242.2)

$$= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right)(\partial_\nu \phi_i) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\mu(\partial_\nu \phi_i)$$

$$= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} (\partial_\nu \phi_i) \right) = \partial_\nu \mathcal{L} = \delta_\nu{}^\mu \partial_\mu \mathcal{L}$$

$$↳ \partial_\mu \Big(\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} (\partial_\nu \phi_i) - \delta_\nu{}^\mu \mathcal{L}}_{\equiv T^\mu{}_\nu} \Big) = 0 \qquad (246.1)$$

como $T^{\alpha\beta} = g^{\beta\nu} T^\alpha{}_\nu = g^{\beta\nu} \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_i)} (\partial_\nu \phi_i) - g^{\beta\nu} \delta_\nu{}^\alpha \mathcal{L}$

↳ $T^{\alpha\beta} = \dfrac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_i)} (\partial^\beta \phi_i) - g^{\alpha\beta}\mathcal{L}$: tensor de tensão canônico (247.1)

notar: Eq. (246.1) → $T^{\alpha\beta}$ satisfaz uma lei de conservação, pois

$$\partial_\alpha T^{\alpha\beta} = 0 \qquad (247.2)$$

ou, em forma integral,

Eq. (64.2)

$$\int_V d^3r\, \partial_0 T^{0\beta} = \frac{1}{c}\frac{d}{dt}\int_V d^3r\, T^{0\beta} = -\int_V d^3r\, \partial_i T^{i\beta} = -\oint_S T^{i\beta}(n_i ds)$$

↳ $\dfrac{1}{c}\dfrac{d}{dt}\displaystyle\int_V d^3r\, T^{0\beta} = -\oint_S T^{i\beta}(n_i ds')$: comparar Eq. (64.2)! (247.3)

↳

próxima etapa: considerar $T^{\alpha\beta}$ p/ caso EM; discussão em duas etapas.

(i) ausência de fontes

nesse caso: $\mathcal{L} = -\dfrac{1}{16\pi} F_{\alpha\beta}F^{\alpha\beta}$: Eq. (243.3)

e $T^{\alpha\beta} = \dfrac{\partial \mathcal{L}}{\partial(\partial_\alpha A^\lambda)}(\partial^\beta A^\lambda) - g^{\alpha\beta}\mathcal{L}$

$$= g_{\alpha\mu}\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\lambda)}}_{= -1/4\pi\, F_{\mu\lambda}}(\partial^\beta A^\lambda) - g^{\alpha\beta}\mathcal{L}$$

Eq. (245.1.0)

$$= -\frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda}(\partial^\beta A^\lambda) - g^{\alpha\beta}\mathcal{L} \qquad (247.4)$$

verifica-se que (veja, e.g., Eqs. (12.105), Jackson) que tensor $T^{\alpha\beta}$ definido em (247.4) não é simétrico ou antisimétrico w.r.t. $\alpha \leftrightarrow \beta$!

↳ próxima etapa: definir tensor simétrico w.r.t. $\alpha \leftrightarrow \beta$ a partir de $T^{\alpha\beta}$ e que preserve a lei de conservação (247.2).

Obs.: $T^{\alpha\beta} \neq$ tensor de tensão de Maxwell $T_{ij}^{(M)}$ definido em (64.1)!

como $F^{\beta\lambda} = \partial^\beta A^\lambda - \partial^\lambda A^\beta$, temos que

$$T^{\alpha\beta} = -\frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda} \Big(\underbrace{F^{\beta\lambda}}_{= -F^{\lambda\beta}} + \partial^\lambda A^\beta \Big) + \frac{1}{16\pi} g^{\alpha\beta} F_{\mu\nu} F^{\mu\nu}$$

$$= \underbrace{\frac{1}{4\pi} \left(g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{4} g^{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \right)}_{\equiv \Theta^{\alpha\beta}} \underbrace{- \frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda} (\partial^\lambda A^\beta)}_{\equiv T_D^{\alpha\beta}}$$

verifica-se que (veja pg. 248.1) $\Theta^{\alpha\beta}$ é simétrico w.r.t. $\alpha \leftrightarrow \beta$ e invariante sob transf. de gauge.

sobre o termo $T_D^{\alpha\beta}$,

$$T_D^{\alpha\beta} = -\frac{1}{4\pi} F^\alpha{}_\lambda (\partial^\lambda A^\beta) = -\frac{1}{4\pi} F^\alpha{}_\lambda g^{\lambda\mu} (\partial_\mu A^\beta) = +\frac{1}{4\pi} F^{\mu\alpha} (\partial_\mu A^\beta)$$

como eq. de Maxwell (213.4): $\frac{1}{4\pi} \partial_\mu F^{\mu\alpha} = \frac{1}{c} J^\alpha = 0$: ausência fontes

↳ $T_D^{\alpha\beta} = \frac{1}{4\pi} \left(F^{\mu\alpha} (\partial_\mu A^\beta) + (\partial_\mu F^{\mu\alpha}) A^\beta \right) = \frac{1}{4\pi} \partial_\mu \left(F^{\mu\alpha} A^\beta \right)$

· propriedades tensor $\Theta^{\alpha\beta}$,

· notar : $g^{\beta\mu} F_{\mu\lambda} F^{\lambda\alpha} = g^{\beta\mu} F_{\mu\lambda} g^{\lambda\bar{\lambda}} g^{\alpha\bar{\mu}} F_{\bar{\lambda}\bar{\mu}}$

$= g^{\alpha\bar{\mu}} g^{\beta\mu} g^{\lambda\bar{\lambda}} F_{\mu\lambda} F_{\bar{\lambda}\bar{\mu}} = g^{\alpha\bar{\mu}} F^{\beta\bar{\lambda}} F_{\bar{\lambda}\bar{\mu}}$

$= g^{\alpha\bar{\mu}} F_{\bar{\mu}\bar{\lambda}} F^{\bar{\lambda}\beta}$: simétrico w.r.t. $\alpha \leftrightarrow \beta$

· Lembrar transf. de gauge (51.3) :

$$\Phi' = \Phi - \frac{1}{c}\partial_t \psi$$

$$\vec{A}' = \vec{A} + \vec{\nabla}\psi$$

$\rightarrow A'^{\alpha} = A^{\alpha} - \partial^{\alpha}\psi$: forma covariante

↳ $F'^{\alpha\beta} = \partial^{\alpha} A'^{\beta} - \partial^{\beta} A'^{\alpha} = \partial^{\alpha}(A^{\beta} - \partial^{\beta}\psi) - \partial^{\beta}(A^{\alpha} - \partial^{\alpha}\psi)$

$= F^{\alpha\beta}$: tensor $F^{\alpha\beta}$ invariante sob transf. gauge

↳ " $\Theta^{\alpha\beta}$ " " " " !

$\hookrightarrow \partial_\alpha T_D^{\alpha\beta} = \frac{1}{4\pi} \partial_\alpha \partial_\mu (F^{\mu\alpha} A^\beta) = \frac{1}{8\pi} \Big(\partial_\alpha\partial_\mu (F^{\mu\alpha} A^\beta) + \underbrace{\partial_\alpha\partial_\mu (F^{\mu\alpha} A^\beta)}\Big)$

$\alpha \leftrightarrow \mu : \quad \partial_\mu \partial_\alpha (\underbrace{F^{\alpha\mu}}_{= -F^{\mu\alpha}} A^\beta)$

$\hookrightarrow \partial_\alpha T_D^{\alpha\beta} = 0$

$\hookrightarrow \partial_\alpha \Theta^{\alpha\beta} = \partial_\alpha (T^{\alpha\beta} - T_D^{\alpha\beta}) = 0$, i.e., $\Theta^{\alpha\beta}$ é simétrico w.r.t. $\alpha \leftrightarrow \beta$ e preserva lei de conservação (247.2) ! (249.1)

definição:

$$\Theta^{\alpha\beta} = \frac{1}{4\pi}\Big(g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{4} g^{\alpha\beta} F_{\mu\nu}F^{\mu\nu}\Big) \quad (249.2)$$

: tensor de tensão simétrico

verifica-se que (veja pg. 249.1):

$$\Theta^{00} = \frac{1}{8\pi}(E^2 + B^2) = u$$

$$\Theta^{0i} = \frac{1}{4\pi}(\vec{E}\times\vec{B})_i = \frac{1}{c}\cdot\frac{c}{4\pi}(\vec{E}\times\vec{B})_i = \frac{1}{c} S_i = c g_i \quad (249.3)$$

$$\Theta^{ij} = -\frac{1}{4\pi}\Big(E_iE_j + B_iB_j - \tfrac{1}{2}\delta_{ij}(E^2+B^2)\Big) = -T_{ij}^{(M)}$$

ou $\Theta^{\alpha\beta} = \left(\begin{array}{c|c} u & c\vec{g} \\ \hline c\vec{g} & -T_{ij}^{(M)} \end{array}\right)$

notar Eq. (249.3): expressões no sistema Gaussiano p/ densidade de energia u, vetor de Poynting $\vec{S}$ e tensor de tensão de Maxwell $T_{ij}^{(M)}$!

Eqs. (247.3) e (249.1): $\frac{1}{c}\frac{d}{dt}\int_V d^3r\, \Theta^{0\beta} = -\oint_S \Theta^{i\beta} (n_i dA)$

· vamos verificar Eqs. (249.3),

Eq. (234.1) : $F_{\mu\nu} F^{\mu\nu} = -2(E^2 - B^2)$

· $\Theta^{00} = \frac{1}{4\pi}\Big(\underbrace{g^{0\mu} F_{\mu\lambda} F^{\lambda 0}}_{F_{0\lambda} F^{\lambda 0} = E^2} - \frac{1}{2}(E^2 - B^2) \Big) = \frac{1}{8\pi}(E^2 + B^2)$

· $\Theta^{0i} = \frac{1}{4\pi} g^{0\mu} F_{\mu\lambda} F^{\lambda i} = \frac{1}{4\pi} F_{0\lambda} F^{\lambda i}$

$i = 1$: $\Theta^{01} = \frac{1}{4\pi}(E_y B_z - E_z B_y) = \frac{1}{4\pi}(\vec{E} \times \vec{B})_1$

similar : $i = 2$ e 3.

· $\Theta^{ij} = \frac{1}{4\pi}\Big(\underbrace{\underbrace{g^{i\mu} F_{\mu\lambda} F^{\lambda j}}_{g^{ii} F_{i\lambda} F^{\lambda j}}}_{= -F_{i\lambda} F^{\lambda j}} + \frac{1}{4} \underbrace{g^{ij}}_{-1/4\, \delta_{ij}} (E^2 - B^2) \Big)$

Como (exercício) : $F_{i\lambda} F^{\lambda j} = E_i E_j + B_i B_j - \delta_{ij} B^2$

$\hookrightarrow \Theta^{ij} = -T_{ij}^{(M)}$

↳ $\beta = 0$: $\frac{1}{c}\frac{d}{dt}\int d^3r\, \Theta^{00} = \frac{1}{c}\frac{d}{dt}\int d^3r\, u = \frac{1}{c}\frac{d}{dt}E_{EM} =$

$= -\oint_S \Theta^{i0}(n_i dA) = -\frac{1}{c}\oint_S \vec{S}\cdot d\vec{A}$: Eq. (61.3) na ausência fontes!

$\beta = j$: $\frac{1}{c}\frac{d}{dt}\int d^3r\, \Theta^{0j} = \frac{d}{dt}\int d^3r\, g_j = \frac{dP_{EM,j}}{dt} =$

$= -\oint_S \Theta^{ij}(n_i dA) = \oint_S T_{ij}^{(M)}(n_i dA) = \oint_S T_{ji}^{(M)}(n_i dA)$:

: Eq. (64.2) na ausência de fontes!

· sobre a conservação do momento angular,

Eq. (66.2) : $\overleftrightarrow{M} = \overleftrightarrow{T}^{(M)} \times \vec{r}$ ou $M_{ij} = T_{ik}^{(M)} r_l \epsilon_{jkl}$

temos que : $M_{ij} = T_{ik}^{(M)} r_l - T_{il}^{(M)} r_k$, $j \neq l \neq k$

↳ momento angular campo EM pode ser escrito como tensor ordem 3 (veja P 6.10, Jackson)

$$M_{ijk} = T_{ij}^{(M)} r_k - T_{ik}^{(M)} r_j \qquad (250.1)$$

verifica-se que (veja P 12.19, Jackson), a conservação do momento angular pode ser escrita em termos tensor

$$M^{\alpha\beta\gamma} = \Theta^{\alpha\beta}x^{\gamma} - \Theta^{\alpha\gamma}x^{\beta} \qquad (250.2)$$

notar : $\partial_\alpha M^{\alpha\beta\gamma} = \underbrace{(\partial_\alpha \Theta^{\alpha\beta})}_{=0} x^{\gamma} + \Theta^{\alpha\beta}\underbrace{\partial_\alpha x^{\gamma}}_{=\delta^{\gamma}{}_{\alpha}} -$

$$- \underbrace{(\partial_\alpha \Theta^{\alpha\gamma})}_{=0} x^{\beta} - \Theta^{\alpha\gamma}\underbrace{\partial_\alpha x^{\beta}}_{=\delta^{\beta}{}_{\alpha}} = \Theta^{\gamma\beta} - \Theta^{\beta\gamma} = 0 \qquad (250.3)$$

(ii) ⊕ fontes externas,

nesse caso, temos que (caso geral) $\partial_\alpha \Theta^{\alpha\beta} \neq 0$

notar:

$$\partial_\alpha \Theta^{\alpha\beta} = \frac{1}{4\pi}\left(\underbrace{g^{\alpha\mu}\partial_\alpha}_{\partial^\mu}(F_{\mu\lambda}F^{\lambda\beta}) + \frac{1}{4}\underbrace{g^{\alpha\beta}\partial_\alpha}_{\partial^\beta}(F_{\mu\nu}F^{\mu\nu})\right)$$

$$= \frac{1}{4\pi}\left(\underbrace{(\partial^\mu F_{\mu\lambda})}_{=\frac{4\pi}{c}J_\lambda} F^{\lambda\beta} + \underbrace{F_{\mu\lambda}\partial^\mu F^{\lambda\beta}}_{F_{\mu\nu}\partial^\mu F^{\nu\beta}} + \underbrace{\frac{1}{4}(\partial^\beta F_{\mu\nu})F^{\mu\nu} + \frac{1}{4}F_{\mu\nu}\partial^\beta F^{\mu\nu}}_{\frac{1}{2}F_{\mu\nu}\partial^\beta F^{\mu\nu}}\right)$$

Eq. de Maxwell
(213.4)

$$\hookrightarrow \partial_\alpha \Theta^{\alpha\beta} + \frac{1}{c}F^{\beta\lambda}J_\lambda = \frac{1}{8\pi}F_{\mu\nu}\left(\partial^\mu F^{\nu\beta} + \underbrace{\partial^\mu F^{\nu\beta} + \partial^\beta F^{\mu\nu}}_{=-\partial^\nu F^{\beta\mu} = \partial^\nu F^{\mu\beta}}\right)$$

Eq. de Maxwell (245.1)

$$= \frac{1}{8\pi}F_{\mu\nu}\left(\partial^\mu F^{\nu\beta} + \partial^\nu F^{\mu\beta}\right) = 0$$

simétrico w.r.t. $\mu \leftrightarrow \nu$

antisimétrico " "

$$\hookrightarrow \partial_\alpha \Theta^{\alpha\beta} + \frac{1}{c}F^{\beta\lambda}J_\lambda = 0 \qquad (251.1)$$

notar: $\beta = 0$: $\partial_\alpha \Theta^{\alpha 0} + \frac{1}{c}F^{0\lambda}J_\lambda = 0$

$$\partial_0 \Theta^{00} + \partial_i \Theta^{i0} + \frac{1}{c}F^{0i}J_i = 0$$

$\hookrightarrow \frac{1}{c}\partial_t u + \frac{1}{c}\vec{\nabla}\cdot\vec{S} + \frac{1}{c}(-\vec{E})\cdot(-\vec{J}) = 0$: Eq. (60.4)

$\beta = i$: $\partial_0 \Theta^{0i} + \partial_j \Theta^{ji} + \frac{1}{c} F^{i\lambda} J_\lambda = 0$

$$\frac{1}{c}\partial_t(cg_i) - \partial_j T_{ij}^{(M)} + \underbrace{\frac{1}{c}F^{i0}J_0}_{1/c\, E_i(c\rho)} + \frac{1}{c}F^{ij}J_j = 0$$

notar: $F^{1j}J_j = F^{12}J_2 + F^{13}J_3 = (-B_z)(-J_y) + (B_y)(-J_z) = (\vec{J}\times\vec{B})_1$

$\hookrightarrow \partial_t g_i + \rho E_i + \frac{1}{c}(\vec{J}\times\vec{B})_i = \partial_j T_{ij}^{(M)}$

ou, em forma integral,

$$\frac{d}{dt}\int_V d^3r\, \vec{g} + \underbrace{\int_V d^3r \left(\rho\vec{E} + \frac{1}{c}\vec{J}\times\vec{B}\right)}_{d\vec{P}_{MEC}/dt} = \int_V d^3r\, \vec{\nabla}\cdot\overleftrightarrow{T} \quad : \text{Eq. } (63.4)$$

· Hamiltoniana do campo EM,

inicial : revisão formalismo Hamiltoniano p/ meios contínuos (p/ detalhes, veja Secs. 13.4, Goldstein, 3ª ed.)

considerar : sistema discreto 1-D, N-partículas massas $\underline{m}$ ⊕ molas, pg. 239

Eq. (222.2) : $p_i = \frac{\partial L}{\partial \dot{q}_i} \longrightarrow \pi_i = \frac{\partial L}{\partial \dot{\eta}_i} = a\frac{\partial L_i}{\partial \dot{\eta}_i}$: momento canonicamente conjugado à $\eta_i = \eta_i(t)$ (252.1)

Eq. (222.3) : $H = \dot{q}_i p_i - L$

$$\hookrightarrow H = \dot{\eta}_i \pi_i - L = a \frac{\partial L_i}{\partial \dot{\eta}_i} \dot{\eta}_i - a \sum_i L_i =$$

$$= \sum_i a \left(\dot{\eta}_i \frac{\partial L_i}{\partial \dot{\eta}_i} - L_i \right) \xrightarrow{a \to 0} H = \int dx \underbrace{\left(\dot{\eta} \frac{\partial \mathcal{L}}{\partial \dot{\eta}} - \mathcal{L} \right)}_{\equiv \mathcal{H}} \quad (253.1)$$

temos que,

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\eta}}$$: densidade momento canonicamente conjugado a $\eta = \eta(x,t)$

e (253.2)

$$\mathcal{H} = \pi \dot{\eta} - \mathcal{L}$$: densidade de Hamiltoniana

Eqs. (253.2) podem ser generalizadas p/ caso sistema descrito campos $\phi_i = \phi_i(\vec{r},t)$:

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} \quad \text{e} \quad \mathcal{H} = \pi^i \dot{\phi}_i - \mathcal{L} \quad (253.3)$$

e, verifica-se que, as eqs. de movimento, equivalentes Eqs. (222.4), são dadas por

$$\frac{\partial \mathcal{H}}{\partial \pi^i} = \partial_t \phi_i \quad \text{e} \quad \frac{\partial \mathcal{H}}{\partial \phi_i} - \frac{d}{dx^j} \left(\frac{\partial \mathcal{H}}{\partial(\partial_j \phi_i)} \right) = -\dot{\pi}^i \quad (253.4)$$

· caso particular : campo EM,

novamente : $\phi_i(\vec{r},t) \to A^\alpha(\vec{r},t)$

(veja pg. 242) $\partial_j \phi_i \to \partial_\mu A^\alpha$

temos que,

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_t A^\mu)} = \frac{1}{c}\frac{\partial \mathcal{L}}{\partial(\partial^0 A^\mu)} \qquad (254.1)$$

$$\mathcal{H} = \pi_\mu \partial_t A^\mu - \mathcal{L} = c\,\pi_\mu \partial^0 A^\mu - \mathcal{L} \qquad (254.2)$$

e $$\frac{\partial \mathcal{H}}{\partial \pi_\mu} = \partial_t A^\mu \qquad \frac{\partial \mathcal{H}}{\partial A^\mu} - \partial_j\left(\frac{\partial \mathcal{H}}{\partial(\partial_j A^\mu)}\right) = -\partial_t \pi_\mu$$

(254.3)

· momentos canonicamente conjugados:

$$\mu = 0 \qquad \pi_0 = \frac{1}{c}\frac{\partial \mathcal{L}}{\partial(\partial^0 A^0)} = 0$$

$$\mu = i \qquad \pi_\mu = \frac{1}{c}\frac{\partial \mathcal{L}}{\partial(\partial^0 A^\mu)} = -\frac{1}{4\pi c} F_{0\mu}$$

como $$F_{0\mu} = g_{0\alpha} g_{\mu\beta} F^{\alpha\beta} = g_{\mu\beta} F^{0\beta} \underset{\mu=i}{\overset{\uparrow}{=}} -F^{0\mu} = -(\partial^0 A^\mu - \partial^\mu A^0)$$

$$= \partial^\mu A^0 - \partial^0 A^\mu = -\partial_i \Phi - \frac{1}{c}\partial_t A_i$$

$$\hookrightarrow \pi_\mu = -\pi_i = -\frac{1}{4\pi c}\left(-\partial_i \Phi - \frac{1}{c}\partial_t A_i\right)$$

$$\hookrightarrow \vec{\pi} = \frac{-1}{4\pi c}\left(\vec{\nabla}\Phi + \frac{1}{c}\partial_t \vec{A}\right) \qquad (254.4)$$

· Hamiltoniana:

Eqs. (245.2) e (254.2),

$$\mathcal{H} = \underbrace{\pi_0 \partial_t \Phi}_{=0} - \pi_i \partial_t A_i - \mathcal{L}$$

notar sinal!

$$\mathcal{H} = -\pi_i \partial_t A_i - \frac{1}{8\pi}\left(\left(-\vec{\nabla}\Phi - \frac{1}{c}\partial_t \vec{A}\right)^2 - (\vec{\nabla}\times\vec{A})^2 \right) + \rho\Phi - \frac{1}{c}\vec{J}\cdot\vec{A}$$

como $\partial_t A_i = -4\pi c^2 \pi_i - c\partial_i \Phi$

$$\hookrightarrow \mathcal{H} = \pi_i\left(4\pi c^2 \pi_i + c\partial_i\Phi\right) - 2\pi c^2 \vec{\pi}^2 + \frac{1}{8\pi}(\vec{\nabla}\times\vec{A})^2 + \rho\Phi - \frac{1}{c}\vec{J}\cdot\vec{A}$$

$$\hookrightarrow \mathcal{H} = 2\pi c^2 \vec{\pi}^2 + \frac{1}{8\pi}(\vec{\nabla}\times\vec{A})^2 + c\,\vec{\pi}\cdot\vec{\nabla}\Phi + \rho\Phi - \frac{1}{c}\vec{J}\cdot\vec{A} \quad : \qquad (255.1)$$

: densidade de Hamiltoniana campo EM !

notar: $\mathcal{H} = \mathcal{H}(\pi_i, A^\mu, \partial_j A^\mu)$

· eqs. de movimento (254.3):

$$\cdot\ \mu = 0: \quad \frac{\partial\mathcal{H}}{\partial\Phi} - \partial_j\left(\frac{\partial\mathcal{H}}{\partial(\partial_j\Phi)}\right) = -\partial_t\pi_0 = 0 \qquad (255.2)$$

$$\cdot\ \mu = i: \quad -\frac{\partial\mathcal{H}}{\partial\pi_i} = \partial_t A_i \qquad (255.3)$$

notar sinais!

$$\frac{\partial\mathcal{H}}{\partial A_i} - \partial_j\left(\frac{\partial\mathcal{H}}{\partial(\partial_j A_i)}\right) = +\partial_t\pi_i \qquad (255.4)$$

temos que:

$$\text{Eq. (255.2)}: \quad \rho - \partial_j(c\pi_j) = 0 \ \rightarrow\ \vec{\nabla}\cdot\vec{\pi} = \frac{1}{c}\rho \qquad (255.5)$$

$$\text{Eq. (255.3)}: \quad -\left(4\pi c^2\pi_i + c\partial_i\Phi\right) = \partial_t A_i$$

$$\hookrightarrow 4\pi c\,\vec{\pi} = -\vec{\nabla}\Phi - 1/c\,\partial_t\vec{A} \qquad (255.6)$$

como $(\vec{\nabla}\times\vec{A})^2 = \epsilon_{k\ell m}\epsilon_{kpq}(\partial_\ell A_m)(\partial_p A_q) =$

$= (\partial_\ell A_m)(\partial_\ell A_m) - (\partial_\ell A_m)(\partial_m A_\ell)$

Eq. (255.4) : $-\frac{1}{c} J_i - \partial_j\left(\partial_j A_i - \partial_i A_j\right)\frac{1}{4\pi} = \partial_t \pi_i$

$$\underbrace{\partial_j\left(\partial_i A_j - \partial_j A_i\right)}_{(\vec{\nabla}\times(\vec{\nabla}\times\vec{A}))_i} - 4\pi\partial_t\pi_i = \frac{4\pi}{c} J_i$$

↳ $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) - 4\pi\partial_t\vec{\pi} = \frac{4\pi}{c}\vec{J}$ (256.1)

notar:

· $\vec{\nabla}\times$ Eq. (255.6) : $\vec{\nabla}\times(4\pi c\,\vec{\pi}) = -1/c\,\partial_t(\vec{\nabla}\times\vec{A})$ (256.2)

· $\vec{\nabla}\cdot$ Eq. (256.2) : $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$ (256.3)

notar : identificando $\vec{E} = 4\pi c\,\vec{\pi}$

↳ Eqs. (255.5), (256.1) - (256.3) : eqs. de Maxwell !

Obs.: no formalismo Hamiltoniano
$\partial_t \phi_i$ → substituído por π_i
mas $\partial_j \phi_i$ preservado, i.e., temos uma assimetria no tratamento das variáveis tempo e espaço

↳ formalismo Hamiltoniano não é adequado p/ uma formulação covariante !