FI 001 – Mecânica Quântica I – Lista 1

01. P.7.2 and P.7.3, Messiah; Ex.9.11 and Ex.9.12, Merzbacher :
Projectors.

a) Prove that projection operators are positive definite.

b) Show that a projection operator has no inverse (if the vector space has more than one dimension).

c) P_1, P_2, ... P_K being projectors, show that their sum is likewise a projector if, and only if

$$\sum_i^k \langle u|P_i|u\rangle \leq \langle u|u\rangle$$

for any vector $|u\rangle$ of Hilbert space.

d) An observable A possesses a finite number N of eigenvalues. One denotes them by a_1, a_2, ..., a_N and sets

$$f(A) \equiv (A - a_1)(A - a_2)\dots(A - a_N) \equiv (A - a_n)g_n(A).$$

Show that $f(A) = 0$ and that the projector P_n upon the subspace of the nth eigenvalue is given by the expression

$$P_n = \frac{g_n(A)}{g_n(a_n)}.$$

P.02. P.7.7, P.7.8, and P.7.9, Messiah:
Positive definite operator.

a) Let H be a positive definite and Hermitean operator. Show that for any $|u\rangle$ and $|v\rangle$,

$$|\langle u|H|v\rangle|^2 \leq \langle u|H|u\rangle\langle v|H|v\rangle$$

and that the equality $\langle u|H|v\rangle = 0$ necessarily implies $H|v\rangle = 0$. Show also that $\mathrm{Tr}H \geq 0$ and that the equality implies $H = 0$.

b) Show that if H and K are two positive definite observables, $\mathrm{Tr}(HK) \geq 0$ and that the equality implies $HK = 0$.

c) A being a linear operator, show that $A^\dagger A$ is a positive definite Hermitean operator and that its trace is equal to the sum of the squares of the moduli of the matrix elements representing A in an arbitrarily chosen representation. Deduce that $\mathrm{Tr}A^\dagger A \geq 0$ and that the equality $\mathrm{Tr}A^\dagger A = 0$ implies $A = 0$.

P.03. P.7.5, Messiah:
Show that

a) in order that a transformation conserve complex conjugation between matrices, it is necessary and sufficient that the transformation matrix be real.

b) in order that a transformation conserve the transposition relation between matrices, it is necessary and sufficient that the transformation matrix be orthogonal.

P.04. Ex.3.12, Ex.3.15, Ex.3.16, and Ex.10.7, Merzbacher:
Commutator algebra.

a) If A and B are two operators that both commute with their commutator $[A, B]$, prove that, for a positive integer n,

$$[A, B^n] = nB^{n-1}[A, B],$$

$$[A^n, B] = nA^{n-1}[A, B].$$

Note the similarity of the process with differentiation. Apply to the special case $A = x$ and $B = p_x$. Prove that

$$[p_x, f(x)] = -i\hbar \frac{df}{dx}\hat{1},$$

if $f(x)$ can be expanded in a power series of the operator x.

b) If $[A, B] = \gamma B$, where γ is a constant, show that

$$e^{\lambda A} B e^{-\lambda A} = e^{\lambda\gamma} B.$$

c) Consider the product $G(\lambda) = e^{\lambda A} e^{\lambda B}$ and prove that

$$\frac{dG}{d\lambda} = \left(A + B + \frac{\lambda}{1!}[A, B] + \frac{\lambda^2}{2!}[A, [A, B]] + \ldots \right) G$$

Show that if A and B are two operators that both commute with their commutator $[A, B]$, then

$$e^A e^B = e^{A+B+[A,B]/2}.$$

d) Consider that A and B are two Hermitian operators that do not commute and that the commutator between A and B assumes the form

$$[A, B] = iC.$$

Show that C is an Hermitian operator.

05. Cap. 5, Messiah:
Prove Eqs. (V.67) and (V.68):

$$[p_i, A] = -i\hbar \frac{\partial A}{\partial q_i},$$

$$[q_i, A] = +i\hbar \frac{\partial A}{\partial p_i}.$$

06. Ex.9.6, Ex.9.7, and Ex.9.10, Merzbacher:

a) Prove that $P_a = \Psi_a(\Psi_a, \ldots)$ is a linear operator.

b) Show that for a linear operator

$$(\Psi_b, A\Psi_a) = \sum_{ij} b_i^* A_{ij} a_j$$

and write this equation in matrix form. If A were antilinear, how would the corresponding equation look?

c) If A is a linear operator, show that it is generally not possible to define a ("transpose") linear operator A^T, which satisfies the equation

$$(\Psi_b, A\Psi_a) = \left(\Psi_a, A^T\Psi_b\right).$$

Show, however, that the above equation defines an antilinear operator A^T, if A is itself antilinear.

07. Ex.9.15, Ex.9.18, and Ex.9.19, Merzbacher and P.7.6, Messiah:

a) For an arbitrary normalized state $|a\rangle$ and an operator A, calculate the sum $\sum_i |\langle a|A|K_i\rangle|^2$ over the entire basis $|K_i\rangle$. What value is obtained if A is unitary?

b) Show by the use of the bra-ket notation that

$$\mathrm{Tr}A = \sum_i \langle u_i|A|u_i\rangle$$

is independent of the choice of the basis $|u_i\rangle$ and that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

c) Show that

$$\sum_{i,j} |\langle u_i|A|v_j\rangle|^2 = \mathrm{Tr}(AA^\dagger) = \mathrm{Tr}(A^\dagger A)$$

and that this expression is independent of the bases $|u_i\rangle$ and $|v_j\rangle$.

d) Let $|u\rangle$ and $|v\rangle$ be two vectors of finite norm. Show that

$$\mathrm{Tr}|u\rangle\langle u| = \langle u|u\rangle,$$

$$\mathrm{Tr}|u\rangle\langle v| = \langle v|u\rangle.$$

P.08. Ex.10.4, Merzbacher:
Prove the identity

$$\det e^A = \exp(\mathrm{Tr}A)$$

directly from the expansion of e^A without recourse to the eigenvalues.

P.09. P.1.1, Desai:
Define de following two state vectors as column matrices:

$$|\alpha_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad |\alpha_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with their Hermitian conjugates given by

$$\langle\alpha_1| = [1\ 0] \qquad \text{and} \qquad \langle\alpha_2| = [0\ 1]$$

respectively. Show the following for $i, j = 1, 2$:

a) The vectors $|\alpha_i\rangle$ are orthonormal.

b) Any column matrix

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

can be written as a linear combination of the vectors $|\alpha_i\rangle$.

c) The outer products $|\alpha_i\rangle\langle\alpha_j|$ form 2×2 matrices which can serve as operators.

d) The vectors $|\alpha_i\rangle$ satisfy the completeness relation

$$\sum_i |\alpha_i\rangle\langle\alpha_i| = \mathbf{1}_{2\times 2}.$$

e) Write the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

as a linear combination of the four matrices formed by $|\alpha_i\rangle\langle\alpha_j|$.

f) Determine the matrix elements of A such that $|\alpha_1\rangle$ and $|\alpha_2\rangle$ are simultaneously the eigenstates of A satisfying the relations

$$A|\alpha_1\rangle = +|\alpha_1\rangle \qquad \text{and} \qquad A|\alpha_2\rangle = -|\alpha_2\rangle.$$

g) The representation of an operator B in the $|\alpha_1\rangle$, $|\alpha_2\rangle$ basis is given by

$$B = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}.$$

Determine the representation of the operators $B^{1/2}$ and $\ln(B)$ in the $|\alpha_1\rangle$, $|\alpha_2\rangle$ basis.

10. P.1.3, Desai:
Show that an unitary operator U can be written as

$$U = \frac{1+iK}{1-iK},$$

where K is a Hermitian operator. Show that one can also write

$$U = \exp(iC),$$

where C is a Hermitian operator. If

$$U = A + iB,$$

show that A and B commute. Express these matrices in terms of C. You can assume that

$$e^M = 1 + M + \frac{1}{2!}M^2 + \ldots,$$

where M is an arbitrary matrix.

P.11. P.1.23, Sakurai:
Consider a three-dimensional ket space. If a certain set of orthonormal kets $|1\rangle, |2\rangle, |3\rangle$ are used as the base kets, the operators A and B are represented by

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{bmatrix},$$

where a and $b \in \Re$.

a) Does operator B exhibit a degenerate spectrum?

b) Show that A and B commute.

c) Find a new set of orthonormal kets which are simultaneously eigenkets of both A and B. Specify the eigenvalues of A and B for each of the three eigenkets. Does your specification of eigenvalues completely characterize each eigenket?

12. P.3.4, Weinberg:
Suppose that a linear operator A, though not Hermitian, satisfies the condition that it commutes with its adjoint. What can be said about the relation between the eigenvalues of A and of $A^\dagger$? What can be said about the scalar product of two eigenstates of A with unequal eigenvalues?

13. Consider two operators A and B that are related by an unitary transformation U, i.e, $B = UAU^\dagger$.

a) Show that $\mathrm{Tr}A = \mathrm{Tr}B$ and $\det A = \det B$.

b) Show that algebraic relations between operators of the form

$$F(A_i) = c_0 + \sum_i c_i A_i + \sum_{i,j} c_{ij} A_i A_j + \ldots = 0$$

are invariant under an unitary transformation, i.e., show that $F(B_i) = 0$.

14. Assume that λ is a small parameter and derive an expansion of the operator $(A - \lambda B)^{-1}$ in powers of λ.

P.15. Unitary operators.

a) Consider an unitary operator U that satisfies the equation $U^2 = U$ and determine its explicit form.

b) Prove that the eigenvalues u_i of a unitary operator U have the property $|u_i|^2 = 1$.

c) Determine the eigenvalues of an operator U that is unitary and Hermitian. Give examples of this kind of operators.

d) Show that the Hermitian and the anti-Hermitian parts of any unitary operator U commutes with each other (consequently, the unitary operator can always be diagonalized). What are the properties of its eigenvalues?

e) Show that an unitary operator U can be written as $U = A + iB$ if the operators A and B commute.

f) Show that an operator of the form $U = \exp(iF)$ is an unitary operator if F is a Hermitian one.

g) Consider that an unitary operator U can be written as $U = A + iB = \exp(iF)$ and determine the operators A and B in terms of F.

16. Ex.2.39, Chuang:
The set L_V of linear operators on a Hilbert space V is obviously a vector space (the sum of two linear operators is a linear operator, zA is a linear operator if A is a linear operator and z is a complex number, and there is a zero element 0). An important additional result is that the vector space L_V can be given a natural inner product structure, turning it into a Hilbert space.

a) Show that the function (,)defined by

$$(A, B) = \mathrm{Tr}(A^\dagger B)$$

is an inner product function. This inner product is known as the Hilbert-Schmidt or trace inner product.

b) If V has d dimensions, show that L_V has dimension d^2.

c) Find an orthonormal basis of Hermitian matrices for the Hilbert space L_V.

P.17. P.2.2, Cohen-Tannoudji:

18. P.2.3, Cohen-Tannoudji:

19. P.2.5, Cohen-Tannoudji:

20. P.2.6, Cohen-Tannoudji:

21. P.2.11, Cohen-Tannoudji:

P.22. P.2.12, Cohen-Tannoudji: