FI 001 – Mecânica Quântica I – Lista 3

P.01. P.12.1 and P.12.2, Messiah:

a) Let a and $a^\dagger$ be two Hermitean conjugate operators such that $[a, a^\dagger] = 1$. We put $N = a^\dagger a$. Show that:
(i) $[N, a^p] = -pa^p$; $[N, (a^\dagger)^p] = +p(a^\dagger)^p$, with $p > 0$ and integer;
(ii) the only algebraic functions of a and $a^\dagger$ which commute with N are the functions of N.

b) Show that the operators a and $a^\dagger$ of Problem 12.1 (above) have no inverse.

P.02. P.12.3, Messiah:
Form the matrices representing the operators q and p in the N representation (notation of Ch. XII.5). Verify that they are Hermitean and satisfy the commutation relation (XII.2).
Set up the eigenvalue problem of q operator in this representation; show that the spectrum of q is nondegenerate continuous and extends from $-\infty$ to $+\infty$. Form explicitly the eigenvector corresponding to the eigenvalue 0.

P.03. Ex.10.13, Ex.10.17, Ex.10.20, and Ex.10.26, Merzbacher:

a) Using the property $D_\alpha^\dagger a D_\alpha = a + \alpha$, show that for any coherent state $|\alpha\rangle$,

$$D_\beta|\alpha\rangle = C|\alpha + \beta\rangle,$$

where $|\alpha + \beta\rangle$ is again a coherent state and C is a phase factor. Interpret this result in terms of the complex eigenvalue plane (Figure 10.1).

b) Prove Eq. (10.111):

$$\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha| = \frac{1}{\pi}\int d\mathrm{Re}\,\alpha\; d\mathrm{Im}\,\alpha\;|\alpha\rangle\langle\alpha| = 1.$$

This is most easily done by expanding the coherent states in terms of the harmonic oscillator eigenstates, using (10.109) and plane polar coordinates in the complex α plane.

c) For a coherent state $|\alpha\rangle$, evaluate the expectation value of the number operator $a^\dagger a$, its square and its variance, using the commutation relation (10.72): $[a, a^\dagger] = 1$. Check the results by computing the expectation values of n, n^2, and $(\Delta n)^2$ directly from the Poisson distribution (10.110).

d) Prove that the operators a and b,

$$a = \lambda b - \nu b^\dagger, \qquad a^\dagger = \lambda b^\dagger - \nu b,$$

are related by a unitary transformation

$$b = U a U^\dagger,$$

where

$$U = \exp\left[\xi\left(a^2 - (a^\dagger)^2\right)/2\right] \qquad \text{and} \qquad e^\xi = \lambda + \nu.$$

Show that U transforms a coherent state into a squeezed state.
Hint: Use identity (3.54).

P.04. P.3.3, Merzbacher:
Consider a linear harmonic oscillator with Hamiltonian

$$H = T + V = \frac{p_x^2}{2m} + \frac{1}{2} m\omega^2 x^2.$$

a) Derive the equation of motion for the expectation value $\langle x \rangle_t$, and show that it oscillates similarly to the classical oscillator as

$$\langle x \rangle_t = \langle x \rangle_0 \cos \omega t + \frac{\langle p_x \rangle_0}{m\omega} \sin \omega t.$$

b) Derive a second-order differential equation of motion for the expectation value $\langle T - V \rangle_t$ by repeated application of Eq. (3.44) and use of the virial theorem. Integrate this equation and, remembering conservation of energy, calculate $\langle x^2 \rangle_t$.

c) Show that

$$\begin{aligned} (\Delta x)_t^2 = \langle x^2 \rangle_t - \langle x \rangle_t^2 &= (\Delta x)_0^2 \cos^2 \omega t + \frac{(\Delta p_x)_0^2}{m^2 \omega^2} \sin^2 \omega t \\ &+ \left[\frac{1}{2}\langle x p_x + p_x x \rangle_0 - \langle x \rangle_0 \langle p_x \rangle_0\right] \frac{\sin 2\omega t}{m\omega}. \end{aligned}$$

Verify that this reduces to the result of Problem 2 (free particle) in the limit $\omega \to 0$.

P.05. P.14.6, Merzbacher:
A linear harmonic oscillator, with energy eigenstates $|n\rangle$, is subjected to a time-dependent interaction between the ground state $|0\rangle$ and the first excited state:

$$V(t) = F(t)|1\rangle\langle 0| + F^*(t)|0\rangle\langle 1|.$$

a) Derive the coupled equations of motion for the probability amplitudes $\langle n|\Psi(t)\rangle$.

b) If $F(t) = \sqrt{2}\hbar\omega\eta(t)$, obtain the energy eigenvalues and the stationary states for $t > 0$.

c) If the system is in the ground state of the oscillator before $t = 0$, calculate $\langle n|\Psi(t)\rangle$ for $t > 0$.

P.06. P.12.6, Messiah:
Let us consider an one-dimensional harmonic oscillator of mass m and angular frequency ω. At the instant $t = 0$, the state of the system is given by

$$|\psi(0)\rangle = \exp\left(\frac{i}{\hbar}p_0\hat{q}\right)\exp\left(-\frac{i}{\hbar}q_0\hat{p}\right)|0\rangle,$$

where $\hat{q}$ and $\hat{p}$ are the position and momentum operators, respectively, q_0 and p_0 are constants, and and $|0\rangle$ is the ground state of the harmonic oscillator.

a) Show that

$$q_0 = \langle\hat{q}\rangle = \langle\psi(0)|\hat{q}|\psi(0)\rangle \qquad \text{and} \qquad p_0 = \langle\hat{p}\rangle = \langle\psi(0)|\hat{p}|\psi(0)\rangle.$$

b) Determine $\langle\hat{q}^2\rangle = \langle\psi(0)|\hat{q}^2|\psi(0)\rangle$ and $\langle\hat{p}^2\rangle = \langle\psi(0)|\hat{p}^2|\psi(0)\rangle$. Show that $|\psi(0)\rangle$ is indeed a minimum wave packet.

c) Show that the wave function corresponding to the state $|\psi(0)\rangle$ is given by

$$\psi(q) = \langle q|\psi(0)\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[\frac{i}{\hbar}\langle p\rangle q - \frac{m\omega}{2\hbar}(q - \langle q\rangle)^2\right].$$

d) With the aid of the identity (XII.29), show that, apart from a phase factor

$$|\psi(t)\rangle = \exp\left(\frac{i}{\hbar}\langle\hat{p}\rangle_t\,\hat{q}\right)\exp\left(-\frac{i}{\hbar}\langle\hat{q}\rangle_t\,\hat{p}\right)|0\rangle,$$

where the average values $\langle\hat{q}\rangle_t = \langle\psi(t)|\hat{q}|\psi(t)\rangle$ of the position operator and $\langle\hat{p}\rangle_t = \langle\psi(t)|\hat{p}|\psi(t)\rangle$ of the momentum operator are respectively given by Eqs. (XII.36) and (XII.37).

e) Determine the coefficients c_n of the expansion of the initial state $|\psi(0)\rangle$ in a series of eigenvectors $|n\rangle$ of the Hamiltonian, i.e., show that

$$|c_n|^2 = e^{-\alpha}\frac{\alpha^n}{n!}, \qquad \alpha = E_{cl}/\hbar\omega, \qquad E_{cl} = \frac{1}{2m}\left(\langle p\rangle^2 + m^2\omega^2\langle q\rangle^2\right).$$

P.07. P.5.6, Cohen-Tannoudji: Charged harmonic oscillator in a variable electric field.

P.08. P.5.7, Cohen-Tannoudji: Translation operator.

Problemas adicionais:

09. P.12.5, Messiah:
Let χ_0, ω_0, and η_0 be the respective initial values of the average values

$$\chi = \langle q^2 \rangle - \langle q \rangle^2, \quad \omega = \langle p^2 \rangle - \langle p \rangle^2, \quad \eta = \langle pq + qp \rangle - 2\langle p \rangle \langle q \rangle$$

relating to a wave packet of an harmonic oscillator. Establish the law of evolution of these mean values as function of time. Show that they are functions of the form $A + B\cos 2\omega t + C\sin 2\omega t$, and that χ and ω remain constant if, and only if

$$\eta_0 = 0, \qquad \omega_0 = m^2\omega^2\chi_0.$$

10. P.10.3, Merzbacher:
Calculate the value of $\Delta x \Delta p$ for a linear harmonic oscillator in its $n-$th energy eigenstate.

11. P.10.7, Merzbacher:
If a coherent state $|\alpha\rangle$ (eigenstate of a) of an oscillator is transformed into a squeezed state by the unitary operator

$$U = \exp\left[\frac{\xi}{2}\left(a^2 - (a^\dagger)^2\right)\right]$$

calculate the value of ξ that will reduce the width of the Hermitian observable $(a + a^\dagger)/\sqrt{2}$ to 1 percent of its original coherent-state value. What happens to the width of the conjugate observable $(a - a^\dagger)/\sqrt{2}i$ in this transformation?

12. P.5.1, Cohen-Tannoudji: harmonic oscillator.

13. P.9.9, Desai:
By expanding the exponential $\exp(ikx)$ in powers of x determine the ground-state expectation value $\langle 0|\exp(ikx)|0\rangle$ and the transition probability amplitude $\langle n|\exp(ikx)|0\rangle$.

14. P.9.11, Desai:
Show that

$$\left[a, F(a^\dagger)\right] = \frac{\partial F}{\partial a^\dagger}, \qquad \text{and} \qquad \left[a^\dagger, F(a)\right] = -\frac{\partial F}{\partial a}$$

by first proving it for $F(b) = b^n$, where $b = a,\ a^\dagger$.

15. P.10.11, Desai:
Determine the propagator function between two coherent states given by

$$\langle\alpha|\exp(-iH_0 t/\hbar)|\beta\rangle,$$

where H_0 is the Hamiltonian of the harmonic oscillator.

16. P.10.13, Desai:
Show that

$$[\exp(ix_0 p/\hbar), a] = x_0\sqrt{m\omega/2\hbar}\,\exp(ix_0 p/\hbar).$$

From this obtain $\langle n|\exp(ix_0 p/\hbar)|0\rangle$.

17. P.12.1, Desai:
From

$$[v_i, v_j] = \frac{iq\hbar}{m^2 c}\epsilon_{ijk}B_k$$

and the Hamiltonian

$$H = \frac{1}{2}m\mathbf{v}^2 = \frac{1}{2m}\left(\mathbf{p} - \frac{1}{c}q\mathbf{A}\right)^2,$$

use the Heisenberg equation of motion to show that if the magnetic field $\mathbf{B}$ is a constant then

$$\frac{d\mathbf{v}}{dt} = \frac{e}{mc}(\mathbf{v}\times\mathbf{B}).$$

Take $\mathbf{B} = B\hat{z}$ and derive the equations for v_x, v_y, and v_z. Show that v_x and v_y satisfy the harmonic oscillator equations while v_z satisfy the free particle equation.

18. P.2.15, Sakurai:
Consider a function, known as the correlation function, defined by

$$C(t) = \langle x(t)x(0)\rangle,$$

where $x(t)$ is the position operator in the Heisenberg picture. Evaluate the correlation function explicitly for the ground state of the 1-D harmonic oscillator.

19. P.2.16, Sakurai:
Consider again a 1-D harmonic oscillator. Do the following algebraically, that is, without using wave functions.

a) Construct a linear combination of $|0\rangle$ and $|1\rangle$ such that $\langle x\rangle$ is as large as possible.

b) Suppose the oscillator is in the state constructed in (a) at $t = 0$. What is the state vector for $t > 0$ in the Schrödinger picture? Evaluate the expectation value $\langle x\rangle$ as a function of time for $t > 0$ using (i) the Schrödinger and (ii) the Heisenberg pictures.

c) Evaluate $\langle(\Delta x)^2\rangle$ as a function of time using either picture.