FI 001 – Mecânica Quântica I – Lista 4

P.01. P.13.2, P.13.3, and P.13.4, Messiah:

a) Show that in any representation where J_x and J_z are real matrices (therefore symmetrical), J_y is a pure imaginary matrix (therefore antisymmetrical).

b) Show that if any operator commutes with two components of an angular momentum (vector) operator, it commutes with the third.

c) Let l be the orbital angular momentum of a particle, θ and ϕ its polar angles and P the parity operator. P is the operator effecting a reflection in the origin; its action on a function $F(\theta, \phi)$ is defined by:

$$PF(\theta, \phi) = F(\pi - \theta, \phi + \pi).$$

Show that $[P, l] = 0$, and from this that the spherical harmonics have a well-defined parity depending only on the quantum number l. Determine it.
Hint: Define $\partial_\theta F(\theta, \phi) = G_1(\theta, \phi)$ and $\partial_\phi F(\theta, \phi) = G_2(\theta, \phi)$.

P.02. P.11.3, Merzbacher:
Explicitly work out the $\mathbf{J}$ matrices for $j = 1/2$, 1, and $3/2$.

P.03. Ex.16.5, Merzbacher, P.7.3, Ballentine, and P.4.3, Weinberg:

a) From the commutation relations, prove that if the z component of some vector operator is represented by a diagonal matrix, J_z must also be diagonal (as must be the z component of any vector operator).

b) Let $\mathbf{A}$ and $\mathbf{B}$ be vector operators. This means that they have certain nontrivial commutation relations with the angular momentum operators. Use those relations to prove that $\mathbf{A} \cdot \mathbf{B}$ commutes with J_x, J_y, and J_z.

c) Suppose that $\mathbf{A}$ and $\mathbf{B}$ are vector operators, in the sense that

$$[J_i, A_j] = i\hbar \sum_k \epsilon_{ijk} A_k \quad \text{and} \quad [J_i, B_j] = i\hbar \sum_k \epsilon_{ijk} B_k$$

Show that the cross-product $\mathbf{A} \times \mathbf{B}$ is a vector in the same sense.

P.04. Ex.11.1 and Ex.16.4, Merzbacher and P.26.3, Desai:

a) The vectors $\mathbf{r}$ and $\mathbf{r}'$ are related via a rotation of an angle θ with respect to the z axis, i.e., $\mathbf{r}' = R_z(\theta)\mathbf{r}$, where the rotation matrix $R_z(\theta)$ is given by Eq. (156.1), Lecture Notes. Show that

$$\mathbf{r}' - \mathbf{r} = \hat{z} \times (\hat{z} \times \mathbf{r})(1 - \cos\phi) + \hat{z} \times \mathbf{r} \sin\phi.$$

Consider now that $\mathbf{r}' = R_{\hat{n}}(\theta)\mathbf{r}$, where $\hat{n}$ is an unit vector, and show that

$$\mathbf{r}' - \mathbf{r} = \hat{n} \times (\hat{n} \times \mathbf{r})(1 - \cos\phi) + \hat{n} \times \mathbf{r} \sin\phi.$$

b) With the aid of Eq. (16.42), Merzbacher, determine the commutators $[\hat{n} \cdot \mathbf{J}, \hat{n} \cdot \mathbf{A}]$ and $[\hat{n} \cdot \mathbf{J}, \hat{n} \times \mathbf{A}]$.

c) Employing the techniques developed in Sec. 3.4, verify that the commutation relations for $\mathbf{A}$ and $\mathbf{J}$ assure the validity of the condition (16.36) or, explicitly, show that

$$\exp\left(i\phi\hat{n} \cdot \mathbf{J}\right) \mathbf{A} \exp\left(-i\phi\hat{n} \cdot \mathbf{J}\right) = \hat{n}(\hat{n} \cdot \mathbf{A}) - \hat{n} \times (\hat{n} \times \mathbf{A}) \cos\phi + \hat{n} \times \mathbf{A} \sin\phi$$

for finite rotations. Note that Eq. (161.3), Lecture Notes is consistent with Eqs. (16.36) and (16.46), Merzbacher.

d) Consider that the vector operators $\mathbf{A}$ and $\mathbf{B}$ are related via a rotation, i.e.,

$$\mathbf{B} = U_{\hat{n}}(\theta)\mathbf{A}U_{\hat{n}}^{\dagger}(\theta),$$

where $\hat{n} = (\hat{x} + \hat{y})/\sqrt{2}$. Determine the components B_x, B_y, and B_z of the operator $\mathbf{B}$ in terms of A_x, A_y, and A_z.

e) For an eigenstate of J_z given by $|j\,m\rangle$ show that for another operator $J_{z'}$ where z' is pointing in the direction making an angle θ with respect to z, the expectation value is $\langle j\,m|J_{z'}|j\,m\rangle = m\cos\theta$.

P.05. P.3.26, Sakurai:
Consider a system with angular momentum $j = 1$.

a) Explicitly write $\langle j = 1\, m' | J_y | j = 1\, m \rangle$ in 3×3 matrix form.

b) Show that

$$\exp\left(-iJ_y\beta/\hbar\right) = 1 - i\left(\frac{J_y}{\hbar}\right)\sin\beta - \left(\frac{J_y}{\hbar}\right)^2 (1 - \cos\beta).$$

c) Using the result from item (b), show that

$$d^{(j=1)}(\beta) = \begin{pmatrix} 1/2(1+\cos\beta) & -1/\sqrt{2}\sin\beta & 1/2(1-\cos\beta) \\ 1/\sqrt{2}\sin\beta & \cos\beta & -1/\sqrt{2}\sin\beta \\ 1/2(1-\cos\beta) & 1/\sqrt{2}\sin\beta & 1/2(1+\cos\beta) \end{pmatrix}.$$

d) Show that for any component $J_u = \hat{n} \cdot \mathbf{J}$, where $\hat{n}$ is an unit operator and one assumes $\hbar = 1$, one has

$$S_u^3 = S_u \qquad \text{and} \qquad \exp(-i\varphi S_u) = 1 - i\sin\varphi S_u - (1 - \cos\varphi)S_u^2.$$

P.06. Ex.17.8 and Ex.17.10, Merzbacher, P.3.9, P.3.20, and P.3.25, Sakurai:

a) Show that the representation matrices $D^{(1/2)}(R)$ are equivalent to the matrices U_R given by Eq. (16.62).

b) Work out the rotation matrices in terms of Euler angles for $j = 0, 1/2$, and 1.

c) Consider a sequence of Euler rotations represented by

$$\mathcal{D}^{(1/2)}(\alpha,\beta,\gamma) = \exp\left(-\frac{i}{2}\alpha\sigma_3\right)\exp\left(-\frac{i}{2}\beta\sigma_2\right)\exp\left(-\frac{i}{2}\gamma\sigma_3\right).$$

Because of the group properties of rotations, we expect that this sequence of operations is equivalent to a single rotation about some axis by an angle θ. Find θ.

d) Consider an orbital angular-momentum eigenstate $|l = 2, m = 0\rangle$. Suppose this state is rotated by an angle β about the y-axis. Find the probability for the new state to be found in $m = 0$, ± 1, and ± 2.

e) Evaluate

$$\sum_{m=-j}^{j} \left| d^{(j)}_{m\,m'}(\beta) \right|^2 m$$

for any j (integer or half-integer); then check your answer for $j = 1/2$.

P.07. P.6.8, Cohen-Tannoudji: Vector operators and rotation.

P.08. P.6.10, Cohen-Tannoudji: Uncertainty relations.

P.09. P.6.11 (a) and (b), Cohen-Tannoudji: Coherent states.

Problemas adicionais:

10. P.6.6, Cohen-Tannoudji: Angular momentum $j = 1$ and rotation.
Hint: Exercise 16.4, Merzbacher.

11. P.9.4, Messiah and P.11.4, Merzbacher:

a) Calculate the commutators of each of component of the $\mathbf{r}$ and $\mathbf{p}$ operators with $\hat{u} \cdot \mathbf{L}$, which is the component of the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ along the unit vector $\hat{u}$. Show that they can be written in the abbreviated form

$$[(\hat{u} \cdot \mathbf{L}), \mathbf{p}] = -i\hbar(\hat{u} \times \mathbf{p}) \qquad \text{and} \qquad [(\hat{u} \cdot \mathbf{L}), \mathbf{r}] = -i\hbar(\hat{u} \times \mathbf{r}).$$

From this, show that every component of $\mathbf{L}$ commutes with the scalar quantities p^2, r^2, and $\mathbf{r} \cdot \mathbf{p}$.

b) Classically, we have for central forces

$$H = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2} + V(r),$$

where $p_r = \mathbf{r} \cdot \mathbf{p}/r$. Show that for translation into quantum mechanics we must write

$$p_r = \frac{1}{2}\left[\frac{1}{r}(\mathbf{r} \cdot \mathbf{p}) + (\mathbf{p} \cdot \mathbf{r})\frac{1}{r}\right]$$

and that this gives the correct Schrödinger equation with the Hermitian operator

$$p_r = -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)$$

whereas $-i\hbar\partial/\partial r$ is not Hermitian.

12. P.13.8, Messiah:
Show that

$$\exp(-i\beta J_y) = \exp(-i\pi J_x/2)\exp(-i\beta J_z)\exp(-i\pi J_x/2).$$

Deduce from this that the matrix elements $\langle j\, m|\exp(-i\beta J_y)|j\, m'\rangle$ are polynomials of degree $2J$ with respect to the variables $\sin\beta/2$ and $\cos\beta/2$.

13. P.13.10, Messiah:
Let s be the intrinsic angular momentum of a particle of spin 1 ($s^2 = s(s+1) = 2$).

a) Show that for any component $S_u \equiv s \cdot u$ one has

$$S_u^3 = S_u \qquad \exp(-i\varphi S_u) = 1 - i\sin\varphi S_u - (1-\cos\varphi)S_u^2$$

and give an explicit expression for the rotation matrix $R(\alpha, \beta, \gamma)$.

b) Let $|z\rangle$ be a vector of norm 1 such that $s_z|z\rangle = 0$, and let $|x\rangle$ and $|y\rangle$ be the vectors obtained from it by rotations of $\pi/2$ about Oy and $-\pi/2$ about Ox respectively. Prove the following relations, as well as those resulting from circular permutation of x, y and z:

$$S_x|x\rangle = 0, \quad S_x|y\rangle = i|z\rangle, \quad S_x|z\rangle = -i|y\rangle, \quad S_x^2|y\rangle = |y\rangle, \quad S_x^2|z\rangle = |z\rangle.$$

Use these to show that $|x\rangle$, $|y\rangle$, and $|z\rangle$ form an orthonormal basis and that the matrices representing S_x, S_y ,and S_z in that basis are those given in Sec. 13.21, Messiah.

c) Show that $\langle i|R(\alpha, \beta, \gamma)|j\rangle = \mathcal{R}_{ij}(\alpha, \beta, \gamma)$ $(i, j = x, y, z)$.
See Secs. 13.10 and 13.14, Messiah for details.

14. Ex. 11.14, Ex.11.15, and Ex.11.21, Merzbacher:

a) Derive L^2 in the coordinate representation from the coordinate representation of L_x, L_y, and L_z.

b) Use the Cartesian representation of

$$L_z = -i\hbar\,(x\partial_y - y\partial_x)$$

to show that $(x \pm iy)^m$ is an eigenfunction of L_z.

c) Check

$$\delta(\mathbf{r} - \mathbf{r}') = \frac{1}{r^2}\delta(r - r')\sum_l \frac{2l+1}{4\pi}P_l(\hat{r}\cdot\hat{r}')$$

by integrating both sides over all of 3-space.

15. P.11.1, Merzbacher:
For the state represented by the wave function

$$\psi = Ne^{-\alpha r^2}(x + y)z.$$

a) Determine the normalization constant N as a function of the parameter α.

b) Calculate the expectation values of $\mathbf{L}$ and L^2.

c) Calculate the variances of these quantities.

16. P.12.2 and P.12.4, Merzbacher:

a) If the ground state of a particle in a spherical square well is just barely bound, show that the well depth V_0 and radius a are related to the binding energy by the expansion

$$\frac{2mV_0a^2}{\hbar^2} = \frac{\pi^2}{4} + 2\kappa a + \left(1 - \frac{4}{\pi^2}\right)(\kappa a)^2 + \dots$$

where $\hbar\kappa = \sqrt{-2mE}$.

b) Show that, if a square well just binds an energy level of angular momentum $l \neq 0$, its parameters satisfy the condition

$$j_{l-1}\left(\sqrt{\frac{2mV_0a^2}{\hbar^2}}\right) = 0.$$

Use recurrence formulas for Bessel functions from standard texts.

Hint: See Sec. 12.3, Merzbacher, for details.

17. P.12.9, Merzbacher:
Solve the Schödinger equation for the three-dimensional isotropic harmonic oscillator, $V(r) = m\omega^2r^2/2$, by separation of variables in Cartesian and in spherical polar coordinates. In the latter case, assume the eigenfunctions to be of the form

$$\psi(r, \theta, \varphi) = r^l \exp\left(\frac{m\omega}{2\hbar}r^2\right) f(r)Y_l^m(\theta, \varphi)$$

and show that $f(r)$ can be expressed as an associated Laguerre polynomial (or a confluent hypergeometric function) of the variable $m\omega r^2/\hbar$ with half-integral indices. Obtain the eigenvalues and establish the correspondence between the two sets of quantum numbers. For the lowest two energy eigenvalues, show the relation between the eigenfunctions obtained by the two methods.

18. P.9.8, Messiah:
Consider the two-dimensional Schrödinger equation for the case where the potential energy $V(r)$ depends only upon the radial variable ($x = r\cos\theta$, $y = r\sin\theta$). Prove the identity

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Deduce from this that there is a complete set of eigenfunctions of the form

$$\psi(r,\theta) = f(r)\exp(il\theta),$$

whose radial part is that solution of the equation

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{l^2}{r^2} + \frac{2m}{\hbar^2}\left(E - V(r)\right)\right] f(r) = 0$$

which vanishes at $r = 0$.

P.19. P.3.13, Sakurai:
An angular-momentum eigenstate $|j, m = j\rangle$ is rotated by an infinitesimal angle ϵ about the y-axis. Without using the explicit form of the $d^{(j)}_{m'\,m}$ function, obtain an expression for the probability for the new rotated state to be found in the original state up to terms of order ϵ^2.

P.20. P.3.17, Sakurai:
The wave function of a particle subjected to a spherically symmetrical potential $V(r)$ is given by

$$\psi(\mathbf{r}) = (x + y + 3z)f(r).$$

a) Is ψ an eigenfunction of L^2? If so, what is the l-value? If not, what are the possible values of l that we may obtain when L^2 is measured?

b) What are the probabilities for the particle to be found in various m_l states?

c) Suppose it is known somehow that $\psi(x)$ is an energy eigenfunction with eigenvalue E. Indicate how we may find $V(r)$.