FI 001 – Mecânica Quântica I – Lista 4

01. Messiah, Cap. 13: 2, 3, 4, 10

02. Merzbacher: Problem 11.4 and Exercise 16.4

03. Cohen-Tannoudji, Cap. 6: 6 (see Exercise 16.4, Merzbacher) and 10

04**. Merzbacher: Problems 12.2 and 12.4 (see Sec. 12.3, Merzbacher)

05. [P 7.3, Ballentine] and [P4.3, Weinberg]

a) Let $\mathbf{A}$ and $\mathbf{B}$ be vector operators. This means that they have certain nontrivial commutation relations with the angular momentum operators. Use those relations to prove that $\mathbf{A} \cdot \mathbf{B}$ commutes with J_x, J_y, and J_z.

b) Suppose that $\mathbf{A}$ and $\mathbf{B}$ are vector operators, in the sense that

$$[J_i, A_j] = i\hbar \sum_k \epsilon_{ijk} A_k \quad \text{and} \quad [J_i, B_j] = i\hbar \sum_k \epsilon_{ijk} B_k$$

Show that the cross-product $\mathbf{A} \times \mathbf{B}$ is a vector in the same sense.

Problemas adicionais:

06. Messiah, Cap. 13: 8

07. Merzbacher, Exercises 11.1, 11.14, 11.15, 11.21, 11.23

08. Merzbacher, Exercises 16.5, 17.8, 17.10 and Problem 12.9

11. [P 26.3, Desai] For an eigenstate of J_z given by $|j\,m\rangle$ show that for another operator $J_{z'}$ where z' is pointing in the direction making an angle θ with respect to z, the expectation value is $\langle j\,m|J_{z'}|j\,m\rangle = m\cos\theta$.
Hint: Exercise 16.4, Merzbacher.

10. [P 3.26, Sakurai]

a) Consider a system with $j = 1$. Explicitly write $\langle j = 1\, m' | J_y | j = 1\, m \rangle$ in 3×3 matrix form.

b) Show that for $j = 1$ only, it is legitimate to replace $\exp(-iJ_y\beta/\hbar)$ by

$$1 - i \left(\frac{J_y}{\hbar} \right) \sin\beta - \left(\frac{J_y}{\hbar} \right)^2 (1 - \cos\beta).$$

c) Using (b), prove that

$$d^{(j=1)}(\beta) = \begin{pmatrix} 1/2(1+\cos\beta) & -1/\sqrt{2}\sin\beta & 1/2(1-\cos\beta) \\ 1/\sqrt{2}\sin\beta & \cos\beta & -1/\sqrt{2}\sin\beta \\ 1/2(1-\cos\beta) & 1/\sqrt{2}\sin\beta & 1/2(1+\cos\beta) \end{pmatrix}$$

12. [P 6.11, Cohen-Tannoudji] Consider a three-dimensional harmonic oscillator, whose state vector $|\psi\rangle$ is

$$|\psi\rangle = |\alpha_x\rangle \otimes |\alpha_y\rangle \otimes |\alpha_z\rangle$$

where $|\alpha_x\rangle$, $|\alpha_y\rangle$, and $|\alpha_z\rangle$ coherent states for one-dimensional harmonic oscillators moving along the axis x, y, and z, respectively. Let $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ be the orbital angular momentum of the three-dimensional oscillator.

a) Prove that:

$$\langle L_z \rangle = i\hbar(\alpha_x \alpha_y^* - \alpha_x^* \alpha_y)$$

$$\Delta L_z = \hbar\sqrt{|\alpha_x|^2 + |\alpha_y|^2}$$

and the analogous expressions for the components of $\mathbf{L}$ along axis x and y.

b) We now assume that:

$$\langle L_x \rangle = \langle L_y \rangle = 0, \qquad \langle L_z \rangle = \lambda\hbar > 0.$$

Show that α_z must be zero. We then fix the value of λ. Show that, in order to minimize $\Delta L_x + \Delta L_y$, we must choose:

$$\alpha_x = -i\alpha_y = \sqrt{\frac{\lambda}{2}} e^{i\varphi_0},$$

where φ_0 is an arbitrary real number. Do the expressions $\Delta J_x \Delta J_y$ and $(\Delta J_x)^2 + (\Delta J_y)^2$ in this case have minimum values which are compatible with the inequalities obtained in item (b) of problem 6.10?