FI 001 – Mecânica Quântica I – Lista 6

P.01. P.15.2, P.15.8, and P.15.10, Messiah:

a) Show that the operator Π defined by Eq. (XV.45),

$$\Pi|\mathbf{r}\,\mu\rangle = |-\mathbf{r}\,\mu\rangle,$$

verifies relations (XV.43) and (XV.44),

$$\Pi\mathbf{r}\Pi^\dagger = -\mathbf{r}, \qquad \Pi\mathbf{p}\Pi^\dagger = -\mathbf{p}, \qquad \Pi\mathbf{S}\Pi^\dagger = \mathbf{S}, \qquad \Pi^2 = 1.$$

b) Show that the translation and rotation operators and the reflection operator Π [definition (XV.45)] commute with the time-reversal operator Θ and that this property does not depend on the choice of the arbitrary phase involved in the definition of Θ.
Hint: Consider infinitesimal translation and rotation transformations.

c) Let K_p be the complex-conjugation operator associated with the $\mathbf{p}$ representation, K_0 the complex-conjugation operator associated with the $\mathbf{r}$ representation, Π the reflection operator [definition (XV.45)]. Show that $K_0 = \Pi K_p$. Deduce that, under time-reversal, the wave function $\Phi(\mathbf{p})$ transforms as

$$K\Phi(\mathbf{p}) = \Phi^*(-\mathbf{p}).$$

P.02. P.4.7, Sakurai and Ex.17.39, Merzbacher:

a) Let $\psi(\mathbf{r}, t)$ be the wave function of a spinless particle corresponding to a plane wave in three dimensions. Show that $\psi^*(\mathbf{r}, -t)$ is the wave function for the plane wave with the momentum direction reversed.

b) Let $\chi(\hat{n})$ be the two-component eigenspinor of $\sigma \cdot \hat{n}$ with eigenvalue $+1$. Using the explicit form of $\chi(\hat{n})$ (in terms of the polar and azimuthal angles θ and φ that characterize $\hat{n}$), verify that $-i\sigma_y\chi^*(\hat{n})$ is the two-component eigenspinor with the spin direction reversed.

c) Show that for a particle of spin one-half, orbital angular momentum l, and total angular momentum j, the eigenstates $\mathcal{Y}_l^{j,m}$ defined in Eq.(17.64) transform under time reversal into $\pm(-1)^m\mathcal{Y}_l^{j,-m}$, the sign depending on whether $j = l - 1/2$ or $j = l + 1/2$.

P.03. P.4.12, Sakurai:
The Hamiltonian for a spin-1 system is given by

$$H = AS_z^2 + B\left(S_x^2 - S_y^2\right).$$

Show that the Hamiltonian is invariant under time reversal. Solve the problem exactly in order to find the normalized energy eigenstates and eigenvalues. Determine the action of the time reversal Θ operator on the eigenvectors considering:

a) the results of Ex.17.37, Merzbacher;

b) the matrix form of the time reversal operator Θ.

P.04. Ex.17.37, Merzbacher:

a) Show that

$$\Theta|j\, m\rangle = e^{i\delta}(-1)^m |j,\, -m\rangle.$$

where $\delta = \delta(j) \in \Re$.

b) Show that

$$\Theta^2|j\, m\rangle = (-1)^{2j}|j\, m\rangle.$$

c) Show that Θ comutes with the rotation operator $U(R) = \exp\left(-i\varphi\hat{n}\cdot\mathbf{J}\right)$.

d) Use the result from the previous item and derive the symmetry relation for the rotation matrices:

$$D^{(j)}_{m',m}(R)^* = (-1)^{m-m'} D^{(j)}_{-m',-m}(R).$$

Hint: see page 250, lecture notes.

P.05. Ex.4.19 and Ex.17.42, Merzbacher; P.13.2 and P.13.5, Ballentine:

a) Show that the Schrödinger equation in the presence of a static magnetic field is not invariant under time reversal.

b) Show that the symmetry operations Θ and Π commute and derive the transformation properties of the fundamental dynamical variables (position, momentum, angular momentum) under the action of the combined inversion-time reversal operation of the antilinear operator $\Theta\Pi$.

c) Show in detail that if $[H, \Pi] = 0$ and if the initial state $|\Psi(0)\rangle$ has definite parity (either even or odd), then the state vector $|\Psi(t)\rangle$ remains a pure parity eigenvector at all future times.

d) Kramer's theorem states that if the Hamiltonian of a system is invariant under time reversal, and if $\Theta^2|\Psi\rangle = -|\Psi\rangle$ (as is the case for an odd number of electrons), then the energy levels must be at least double degenerate. In fact the degree of degeneracy must be even. Show explicitly that threefold degeneracy is not possible.

P.06. Ex.17.41, Merzbacher:
An irreducible spherical tensor operator $T_q^{(k)}$ is said to be even or odd with respect to time-reversal if it satisfies the condition

$$\Theta T_q^{(k)} \Theta^\dagger = \pm(-1)^q T_{-q}^{(k)}.$$

By taking the matrix element of the operator equation above between states of sharp angular momentum, using the results of Ex.17.37, and the antilinearity of Θ, show that

$$\langle \alpha'\, j'\, m'|T_q^{(k)}|\alpha\, j\, m\rangle = \pm e^{i(\delta'-\delta)}\langle \alpha'\, j',\, -m'|T_{-q}^{(k)}|\alpha\, j,\, -m\rangle^*.$$

Consider the diagonal elements in α and j and using the Wigner-Eckart theorem show that

$$\langle \alpha\, j||T^{(k)}||\alpha\, j\rangle = \pm(-1)^k \langle \alpha\, j, ||T^{(k)}||\alpha\, j\rangle^*.$$

Hint: see page 250, lecture notes.

P.07. Ex.4.17, Merzbacher:
Consider the Schrödinger equation in the presence of an electromagnetic field,

$$i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = -\frac{\hbar^2}{2m}\left(\nabla - \frac{iq}{\hbar c}\mathbf{A}\right)^2 \psi(\mathbf{r}, t) + q\phi\psi(\mathbf{r}, t).$$

a) Derive the corresponding continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0,$$

and determine the new current density $\mathbf{j}$.

b) Show that the current density $\mathbf{j}$ obtained above can also be determined from the expression

$$\mathbf{j} = \frac{\hbar}{2mi}\left(\psi^* \nabla \psi - (\nabla \psi^*)\psi\right) = \frac{1}{m}\mathrm{Re}\left[\psi^*\left(-i\hbar \nabla \psi\right)\right]$$

via the substitution $\nabla \to \nabla - \frac{iq}{\hbar c}\mathbf{A}$.

c) Show that this new current density $\mathbf{j}$ is gauge-invariant.

Problemas adicionais:

08. P.15.3, Messiah:
The operator of reflection in a plane perpendicular to $\mathbf{u}$ for a particle of spin $1/2$ is the product of an operator acting on the orbital variables alone with an operator acting on the spin variables alone. Show that for a particular choice of phase factor, the last mentioned operator is equal to $(\sigma \cdot \mathbf{u})$. Show that the product $(\sigma \cdot \mathbf{v})(\sigma \cdot \mathbf{u})$ representing two successive reflections in planes perpendicular to the vectors $\mathbf{u}$ and $\mathbf{v}$ respectively is equal to the operator associated with a rotation of the spin through an angle $2(\mathbf{u}, \mathbf{v})$ about the axis $(\mathbf{u} \times \mathbf{v})/|\mathbf{u} \times \mathbf{v}|$.

09. P.15.11, Messiah:
Suppose that we have a system that is invariant under time-reversal. Let B be a pure imaginary (cf. 19) observable of this system (momentum, spin, orbital angular momentum, etc.).

a) Show that if a stationary state is non-degenerate, the average value of B in that state is null;

b) Show that the trace of the projection of B onto the subspace of any energy eigenvalue is null.

10. Ex.4.14, Merzbacher (Bloch theorem):
If $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3$ are the primitive translation vectors of a three-dimensional infinite lattice whose points are at positions $\mathbf{R} = n_1\mathbf{d}_1 + n_2\mathbf{d}_2 + n_3\mathbf{d}_3$ $(n_i =$ integer$)$, show that any simultaneous eigenfunction of all the translation operators $D_R = \exp(-i\mathbf{k} \cdot \mathbf{R})$ corresponding to eigenvalues $\exp(-i\mathbf{k}' \cdot \mathbf{R})$ is of the form $u(\mathbf{r})\exp(-i\mathbf{k}' \cdot \mathbf{r})$, where $u(\mathbf{r})$ is an arbitrary periodic function of $\mathbf{r}$ over the lattice, i.e., $u(\mathbf{r} + \mathbf{R}) = u(\mathbf{r})$, for every $\mathbf{R}$. Note that any two values of $\mathbf{k}'$ which produce the same eigenvalue $\exp(-i\mathbf{k}' \cdot \mathbf{R})$ differ by a reciprocal lattice vector $\mathbf{G}$, defined by the condition $\mathbf{G} \cdot \mathbf{R} = 2\pi n$, where n is any integer.

11. P.11.10, Ballentine:
The Hamiltonian for a charged particle in a homogeneous magnetic field is

$$H = \frac{1}{2m}\left(-i\hbar\nabla - \frac{q}{c}\mathbf{A}(\mathbf{r})\right)^2,$$

with $\nabla \times \mathbf{A}(\mathbf{r}) = \mathbf{B}$ being a constant. Although the physical situation (a homogeneous magnetic field) is translationally invariant, it is apparent that the operator H is not translationally invariant. Show, however, that the displaced Hamiltonian H' obtained by the translation $T(\mathbf{a})$ is related to H by a gauge transformation.

12. P.4.4, Sakurai:
A spin $1/2$ particle is bound to a fixed center by a spherically symmetrical potential.

a) Write down the spin-angular function $\mathscr{Y}_{l=0}^{j=1/2,m=1/2}$.

b) Express $(\sigma \cdot \mathbf{r})\mathscr{Y}_{l=0}^{j=1/2,m=1/2}$ in terms of some other $\mathscr{Y}_{l}^{j,m}$.

c) Show that your result in (b) is understandable in view of the transformation properties of the operator $\mathbf{S} \cdot \mathbf{r}$ under rotations and under parity.

Hint: Eq. (17.64), Merzbacher.

13. P.4.8, Sakurai:

a) Assuming that the Hamiltonian is invariant under time reversal, prove that the wave function for a spinless nondegenerate system at any given instant of time can always be chosen to be real.

b) The wave function for a plane-wave at $t = 0$ is given by a complex function $e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}$. Why does this not violate time-reversal invariance?

14. P.17.1, Merzbacher:
In the notation of Eq.(17.64), Merzbacher, the state of a spin-$1/2$ particle with sharp total angular momentum j, m is:

$$a\mathscr{Y}_{j-1/2}^{j,m} + b\mathscr{Y}_{j+1/2}^{j,m}.$$

Assume this state to be an eigenstate of the Hamiltonian with no degeneracy other than that demanded by rotation invariance.

a) If H conserves parity, how are the coefficients a and b restricted?

b) If H is invariant under time reversal, show that a/b must be imaginary.

Hint: Exercise 17.39, Merzbacher.

15. The Aharonov-Bohm effect:

a) P.2.28, Sakurai.

b) P.4.3, Merzbacher.

16. P.15.7, Messiah and P.3.7, Ballentine:
The unitary operator $U(\mathbf{v}) = \exp(-i\mathbf{v}\cdot\mathbf{G}/\hbar)$ describes the effect of a transformation to a frame of reference moving at the velocity $\mathbf{v}$ with respect to the original reference frame.

a) An instantaneous ($t = 0$) Galilei transformation is given by the generator $\mathbf{G} = -m\mathbf{r}$, where m is the mass particle. Show that the components of the generator $\mathbf{G}$ satisfy the algebra

$$[G_i, G_j] = 0, \qquad [G_i, r_j] = 0, \qquad [G_i, p_j] = -\imath\hbar m\delta_{i,j}, \qquad [G_i, L_j] = i\hbar\epsilon_{ijk}G_k,$$

and that the position and the momentum operators transform as

$$U(\mathbf{v})\mathbf{r}U^\dagger(\mathbf{v}) = \mathbf{r}, \qquad \text{e} \qquad U(\mathbf{v})\mathbf{p}U^\dagger(\mathbf{v}) = \mathbf{p} - m\mathbf{v}.$$

b) Consider that the Hamiltonian for the particle is given by

$$H = \frac{1}{2m}\big(\mathbf{p} - \mathbf{A}(\mathbf{r})\big)^2 + \Phi(\mathbf{r}).$$

Determine $H'(\mathbf{r},\mathbf{p}) = U(\mathbf{v})H(\mathbf{r},\mathbf{p})U^\dagger(\mathbf{v})$. Express $H' = H'(\mathbf{r}',\mathbf{p}')$ in terms of $\mathbf{r}'$ and $\mathbf{p}'$, and show that the Hamiltonian H is invariant under the instantaneous Galilei transformation $U(\mathbf{v})$.

c) The generator for a complete Galilei transformation is given by $\mathbf{G} = -m\mathbf{r} + \mathbf{p}t$. In this case, determine the algebra that the components of the generator satisfy, i.e., determine the commutators

$$[G_i, G_j], \qquad [G_i, r_j], \qquad [G_i, p_j], \qquad [G_i, L_j],$$

and show that

$$U(\mathbf{v},t)\mathbf{r}U^\dagger(\mathbf{v},t) = \mathbf{r} - \mathbf{v}t \qquad \text{e} \qquad U(\mathbf{v},t)\mathbf{p}U^\dagger(\mathbf{v},t) = \mathbf{p} - m\mathbf{v}.$$

d) Consider that the Hamiltonian for the particle is given by

$$H = \frac{1}{2m}\big(\mathbf{p} - \mathbf{A}(\mathbf{r},t)\big)^2 + \Phi(\mathbf{r},t).$$

Determine $H'(\mathbf{r},\mathbf{p}) = U(\mathbf{v},t)H(\mathbf{r},\mathbf{p})U^\dagger(\mathbf{v},t)$. Express $H' = H'(\mathbf{r}',\mathbf{p}')$ in terms of $\mathbf{r}'$ and $\mathbf{p}'$, and show that the Hamiltonian H is invariant under the instantaneous Galilei transformation $U(\mathbf{v},t)$.