FI 001 – Mecânica Quântica I – Lista 6

01. Messiah, Cap. 15: 2, 8, 10

02. [P 4.12, Sakurai]
The Hamiltonian for a spin-1 system is given by

$$H = AS_z^2 + B\left(S_x^2 - S_y^2\right).$$

Show that the Hamiltonian is invariant under time reversal. Solve the problem exactly in order to find the normalized energy eigenstates and eigenvalues. Determine the action of the time reversal Θ operator on the eigenvectors considering (i) the results of Ex. 17.37, Merzbacher and (ii) the matrix form of the time reversal operator Θ.

03. [Ex. 17.37, Merzbacher]
Show that

$$\Theta|j\,m\rangle = e^{i\delta}(-1)^m|j,\,-m\rangle$$

and

$$\Theta^2|j\,m\rangle = (-1)^{2j}|j\,m\rangle,$$

where $\delta = \delta(j) \in \Re$. Show that Θ comutes with the rotation operator $U(R) = \exp\left(-i\varphi\hat{n}\cdot\mathbf{J}\right)$, and use this information to derive the symmetry relation for the rotation matrices:

$$D^{(j)}_{m',m}(R)^* = (-1)^{m-m'}D^{(j)}_{-m',-m}(R).$$

Hint: see page 250, lecture notes.

04.

a) (Ex. 4.19, Merzbacher) Show that the Schrödinger equation in the presence of a static magnetic field is not invariant under time reversal.

b) (Ex. 17.42, Merzbacher) Show that the symmetry operations Θ and Π commute and derive the transformation properties of the fundamental dynamical variables (position, momentum, angular momentum) under the action of the combined inversion-time reversal operation of the antilinear operator $\Theta\Pi$.

c) (P 13.2, Ballentine) Show in detail that if $[H,\Pi] = 0$ and if the initial state $|\Psi(0)\rangle$ has definite parity (either even or odd), then the state vector $|\Psi(t)\rangle$ remains a pure parity eigenvector at all future times.

d) (P 13.5, Ballentine) Kramer's theorem states that if the Hamiltonian of a system is invariant under time reversal, and if $\Theta^2|\Psi\rangle = -|\Psi\rangle$ (as is the case for an odd number of electrons), then the energy levels must be at least double degenerate. In fact the degree of degeneracy must be even. Show explicitly that threefold degeneracy is not possible.

05. [Ex. 17.41, Merzbacher]
An irreducible spherical tensor operator $T_q^{(k)}$ is said to be even or odd with respect to time-reversal if it satisfies the condition

$$\Theta T_q^{(k)} \Theta^\dagger = \pm(-1)^q T_{-q}^{(k)}.$$

By taking the matrix element of the operator equation above between states of sharp angular momentum, using the results of Ex. 17.37, and the antilinearity of Θ, show that

$$\langle \alpha'\, j'\, m' | T_q^{(k)} | \alpha\, j\, m\rangle = \pm e^{i(\delta-\delta')} \langle \alpha'\, j',\, -m' | T_{-q}^{(k)} | \alpha\, j,\, -m\rangle^*.$$

Consider the diagonal elements in α and j and using the Wigner-Eckart theorem show that

$$\langle \alpha\, j || T^{(k)} || \alpha\, j\rangle = \pm(-1)^k \langle \alpha\, j, || T^{(k)} || \alpha\, j\rangle^*.$$

Hint: see page 250, lecture notes.

06. [Ex. 4.17, Merzbacher]
Consider the Schrödinger equation in the presence of an electromagnetic field,

$$i\hbar \frac{\partial \psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\left(\nabla - \frac{iq}{\hbar c}\mathbf{A}\right)^2 \psi(\mathbf{r},t) + q\phi\psi(\mathbf{r},t),$$

derive the corresponding continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0,$$

and determine the new current density $\mathbf{j}$. Show that the current density $\mathbf{j}$ obtained above can also be determined from the expression

$$\mathbf{j} = \frac{\hbar}{2mi}\left(\psi^* \nabla\psi - (\nabla\psi^*)\psi\right) = \frac{1}{m}\mathrm{Re}\left[\psi^*\left(-i\hbar\nabla\psi\right)\right]$$

via the substitution $\nabla \to \nabla - \frac{iq}{\hbar c}\mathbf{A}$. Finally, show that this new current density $\mathbf{j}$ is gauge-invariant.

Problemas adicionais:

07. Messiah, Cap. 15: 3, 11

08. Merzbacher, Exercises 4.14 (Bloch theorem)

11**. [P 15.7, Messiah and P 3.7, Ballentine]
The unitary operator $U(\mathbf{v}) = \exp(i\mathbf{v} \cdot \mathbf{G})$ describes the effect of a transformation to a frame of reference moving at the velocity $\mathbf{v}$ with respect to the original reference frame.

a) An instantaneous ($t = 0$) Galilei transformation is given by the generator $\mathbf{G} = -m\mathbf{r}$, where m is the mass particle. Show that the components of the generator $\mathbf{G}$ satisfy the algebra

$$[G_i, G_j] = 0, \qquad [G_i, r_j] = 0, \qquad [G_i, p_j] = -\imath\hbar m \delta_{i,j}, \qquad [G_i, L_j] = i\hbar \epsilon_{ijk} G_k,$$

and that the position and the momentum operators transform as

$$U(\mathbf{v})\mathbf{r}U^\dagger(\mathbf{v}) = \mathbf{r}, \qquad U(\mathbf{v})\mathbf{p}U^\dagger(\mathbf{v}) = \mathbf{p} - m\mathbf{v}.$$

b) Consider that the Hamiltonian for the particle is given by

$$H = \frac{1}{2m}\left(\mathbf{p} - \mathbf{A}(\mathbf{r})\right)^2 + \Phi(\mathbf{r}).$$

Determine the commutator $[G_i, H]$ and show that the Hamiltonian H is invariant under the instantaneous Galilei transformation $U(\mathbf{v})$, i.e., determine $H'(\mathbf{r}', \mathbf{p}') = U(\mathbf{v})H(\mathbf{r}, \mathbf{p})U^\dagger(\mathbf{v})$.

c) The generator for a complete Galilei transformation is given by $\mathbf{G} = -m\mathbf{r} + \mathbf{p}t$. In this case, determine the algebra that the components of the generator satisfy and show that

$$U(\mathbf{v}, t)\mathbf{r}U^\dagger(\mathbf{v}, t) = \mathbf{r} - \mathbf{v}t, \qquad U(\mathbf{v}, t)\mathbf{p}U^\dagger(\mathbf{v}, t) = \mathbf{p} - m\mathbf{v}.$$

d) Consider that the Hamiltonian for the particle is given by

$$H = \frac{p^2}{2m} + \Phi(\mathbf{r}).$$

Determine the commutator $[G_i, H]$ and show that the Hamiltonian H is invariant under the complete Galilei transformation $U(\mathbf{v}, t)$.

10. [P 11.10, Ballentine]
The Hamiltonian for a charged particle in a homogeneous magnetic field is

$$H = \frac{1}{2m}\left(-i\hbar\nabla - \frac{q}{c}\mathbf{A}(\mathbf{r})\right)^2,$$

with $\nabla \times \mathbf{A}(\mathbf{r}) = \mathbf{B}$ being a constant. Although the physical situation (a homogeneous magnetic field) is translationally invariant, it is apparent that the operator H is not translationally invariant. Show, however, that the displaced Hamiltonian H', obtained by the transformation $\mathbf{r} \to \mathbf{r} + \mathbf{a}$, is related to H by a gauge transformation.

12. [P 4.4, Sakurai]
A spin $1/2$ particle is bound to a fixed center by a spherically symmetrical potential.

a) Write down the spin-angular function $\mathscr{Y}_{l=0}^{j=1/2,m=1/2}$.

b) Express $(\sigma \cdot \mathbf{r})\mathscr{Y}_{l=0}^{j=1/2,m=1/2}$ in terms of some other $\mathscr{Y}_{l}^{j,m}$.

c) Show that your result in (b) is understandable in view of the transformation properties of the operator $\mathbf{S} \cdot \mathbf{r}$ under rotations and under parity.

Hint: Eq. (17.64), Merzbacher.

13. [P 4.7, Sakurai]

a) Let $\psi(\mathbf{r}, t)$ be the wave function of a spinless particle corresponding to a plane wave in three dimensions. Show that $\psi^*(\mathbf{r}, -t)$ is the wave function for the plane wave with the momentum direction reversed.

b) Let $\chi(\hat{n})$ be the two-component eigenspinor of $\sigma \cdot \hat{n}$ with eigenvalue $+1$. Using the explicit form of $\chi(\hat{n})$ (in terms of the polar and azimuthal angles θ and φ that characterize $\hat{n}$), verify that $-i\sigma_y\chi^*(\hat{n})$ is the two-component eigenspinor with the spin direction reversed.

14. [P 4.8, Sakurai]

a) Assuming that the Hamiltonian is invariant under time reversal, prove that the wave function for a spinless nondegenerate system at any given instant of time can always be chosen to be real.

b) The wave function for a plane-wave at $t = 0$ is given by a complex function $e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}$. Why does this not violate time-reversal invariance?

15. The Aharonov-Bohm effect

a) P 2.28, Sakurai

b) P 4.3, Merzbacher