FI 001 – Mecânica Quântica I – Lista 5

P.01. Ex.16.9, Ex.16.10, Ex.16.11, Ex.16.12, and Ex.16.19, Merzbacher:

a) Take advantage of the properties (16.54) and (16.55) of the Pauli matrices to work out the eigenvalues and eigenspinors of A in terms of the expansion coefficients λ_0, λ_1, λ_2, and λ_3. Specialize to the case $\lambda_0 = 0$ and $\vec{\lambda} = \hat{n}$, where $\hat{n}$ is a real-valued arbitrary unit vector, and show that the eigenvalues are given by

$$\chi_{+/-} = \frac{1}{\sqrt{2}} \begin{pmatrix} (n_x - in_y)/(1 \mp n_z)^{1/2} \\ (1 \mp n_z)^{1/2} \end{pmatrix}.$$

b) Show that the 2×2 matrix

$$U = e^{i\gamma} \left(1 \cos\omega + i\hat{n} \cdot \sigma \sin\omega \right),$$

where γ and ω are real angles, and $\hat{n}$ is a real unit vector, is an unitary matrix.

c) If $\mathbf{A}$ and $\mathbf{B}$ are two vectors that commute with σ, prove the useful identity

$$(\sigma \cdot \mathbf{A})(\sigma \cdot \mathbf{B}) = \mathbf{A} \cdot \mathbf{B} + i\sigma \cdot (\mathbf{A} \times \mathbf{B}).$$

d) Using the special properties of the Pauli matrices, prove directly that

$$U_R^\dagger \sigma U_R = \hat{n}(\hat{n} \cdot \sigma) - \hat{n} \times (\hat{n} \cdot \sigma) \cos\phi + \hat{n} \times \sigma \sin\phi$$

if U_R is given by Eq. (16.62).

e) Given a spinor

$$\chi = \begin{pmatrix} e^{i\alpha} \cos\delta \\ e^{i\beta} \sin\delta \end{pmatrix},$$

calculate the polarization vector $\mathbf{P}$ and construct the matrix U_R which rotates this state into $(1\ 0)^T$. Prove that the probability $p_{\hat{n}}$ of finding this particle to be in a state represented by the polarization vector $\hat{n}$ is

$$p_{\hat{n}} = \frac{1}{2} \mathrm{Tr} \left[\rho (1 + \hat{n} \cdot \sigma) \right] = \frac{1}{2} \left(1 + \mathbf{P} \cdot \hat{n} \right)$$

and show that this result agrees with expectations for $\hat{n} = \mathbf{P}$, $\hat{n} = -\mathbf{P}$, and $\hat{n} \perp \mathbf{P}$.

P.02. P.16.1, Merzbacher:
The spin-zero neutral kaon is a system with two basis states, the eigenstates of σ_z, representing a particle K^0 and its antiparticle $\bar{K}^0$: The operator $\sigma_x = CP$ represents the combined parity (P) and charge conjugation (C), or particle-antiparticle, transformation and takes $|+\rangle = |K^0\rangle$ into $|-\rangle = |\bar{K}^0\rangle$. The dynamics is governed by the Hamiltonian matrix

$$H = M - \frac{i}{2}\Gamma,$$

where M and Γ are Hermitian 2×2 matrices, representing the mass-energy and decay properties of the system, respectively. The matrix Γ is positive definite. A fundamental symmetry (under the combined CP and time reversal transformations) requires that $\sigma_x M^* = M\sigma_x$ and $\sigma_x \Gamma^* = \Gamma\sigma_x$.

a) Show that in the expansion of H in terms of the Pauli matrices, the matrix σ_z is absent. Derive the eigenvalues and eigenstates of H in terms of the matrix elements of M and Γ. Are the eigenstates orthogonal?

b) Assuming, as is the case to good approximation, that the Hamiltonian also satisfies the CP invariance conditions $\sigma_x M = M\sigma_x$ and $\sigma_x \Gamma = \Gamma\sigma_x$, show that H is normal, and construct its eigenstates $|K_1^0\rangle$ and $|K_2^0\rangle$.

c) If the kaon is produced in the state K^0 at $t = 0$, calculate the probability of finding it still to be a K^0 at a later time t.

P.03. P.13.12, Messiah and Ex.17.17, Merzbacher:

a) Consider a spin-$1/2$ particle. Show that in the space of states of a given orbital angular momentum l, the operators

$$\frac{l+1+\mathbf{L}\cdot\sigma}{2l+1} \qquad \text{and} \qquad \frac{l-\mathbf{L}\cdot\sigma}{2l+1}$$

are projectors onto the states of total angular momentum $j = l + 1/2$ and $j = l - 1/2$ respectively.

b) Apply

$$J^2 = L^2 + S^2 + 2\mathbf{L}\cdot\mathbf{S} = L^2 + S^2 + L_+S_- + L_-S_+ + 2L_zS_z$$

to Eq. (17.64) and verify that $\mathcal{Y}_l^{jm}$ is an eigenstate of J^2.

P.04. P.17.4, Merzbacher:
The Hamiltonian of the positronium atom in the 1S state in a magnetic field B along the z axis is to good approximation

$$H = A\mathbf{S}_1 \cdot \mathbf{S}_2 + \frac{eB}{mc}\,(S_{1z} - S_{2z})$$

if all higher energy states are neglected. The electron is labeled as particle 1 and the positron as particle 2. Using the coupled representation in which $\mathbf{S}^2 = (\mathbf{S}_1 + \mathbf{S}_2)^2$ and $S_z = S_{1z} + S_{2z}$ are diagonal, obtain the energy eigenvalues and eigenvectors and classify them according to the quantum numbers associated with the constants of the motion.

P.05. P.10.6 (a) and (b), Cohen-Tannoudji: Two spin $S = 1/2$ particles.

P.06. P.13.14, Messiah:
We denote by $J^2\{A\}$ the following function of the operator A and the components of angular momentum:

$$J^2\{A\} \equiv [J_x, [J_x, A]] + [J_y, [J_y, A]] + [J_z, [J_z, A]].$$

Show that if $T^{(k)}$ is a kth-order irreducible tensor operator, its components verify the relation

$$J^2\{T_q^{(k)}\} = k(k+1)T_q^{(k)}.$$

P.07. P.13.16, Messiah, P.7.11 and P.7.12, Ballentine:

a) Show that if the $(2k+1)$ operators $T_q^{(k)}$ $(q = -k, ..., +k)$ verify the commutation relations (XIII.123), the $(2k+1)$ operators $S_q^{(k)} \equiv (-1)^q T_{-q}^{(k)\dagger}$ have the same property.

b) Prove that the operator defined by

$$T_m^{(j)} = \sum_{q,q'} \langle k\, k'\, q\, q' | k\, k'\, j\, m\rangle U_q^{(k)} V_{q'}^{(k')} \tag{1}$$

is indeed an irreducible tensor operator of rank j. Here, $U_q^{(k)}$ and $V_{q'}^{(k')}$ are respectively irreducible tensor operators of rank k and k' and $\langle k\, k'\, q\, q' | k\, k'\, j\, m\rangle$ are Clebsh-Gordan coefficients.

c) Consider two vector operators $\mathbf{U}$ and $\mathbf{V}$. Evaluate Eq. (1) and show that

$$
\begin{aligned}
T_0^{(0)} &= -\frac{1}{\sqrt{3}}\mathbf{U}\cdot\mathbf{V}, \\
T_m^{(1)} &= \frac{i}{\sqrt{2}}\left(\mathbf{U}\times\mathbf{V}\right)_m, \\
T_{\pm 2}^{(2)} &= U_{\pm 1}V_{\pm 1}, \\
T_{\pm 1}^{(2)} &= \frac{1}{\sqrt{2}}\left(U_{\pm 1}V_0 + U_0 V_{\pm 1}\right), \\
T_0^{(2)} &= \frac{1}{\sqrt{6}}\left(U_1 V_{-1} + 2U_0 V_0 + U_{-1}V_1\right).
\end{aligned}
\tag{2}
$$

P.08. Ex.17.26 and Ex.17.28, Merzbacher:

a) Show that the trace of any irreducible spherical tensor operator vanishes, except those of rank 0 (scalar operators).

b) If S_k^q and T_k^q are two irreducible spherical tensor operators of rank k, prove that the contracted operator

$$\sum_{q=-k}^{+k}(-1)^q S_k^q T_k^{-q}$$

is a tensor operator of rank zero, i.e., a scalar operator. For $k=1$, show that this scalar operator is just the inner product $\mathbf{S}\cdot\mathbf{T}$.

P.09. Ex.17.24, Merzbacher and P.3.32, Sakurai:

a) Derive the angular momentum selection rules for an irreducible second-rank (quadrupole) tensor operator.

b) Write xy, xz, and (x^2-y^2) as components of a spherical (irreducible) tensor of rank 2.

c) The expectation value $Q \equiv e\langle \alpha\, j\, m=j|3z^2-r^2|\alpha\, j\, m=j\rangle$ is known as the quadrupole moment. Evaluate

$$e\langle \alpha\, j\, m'|x^2-y^2|\alpha\, j\, m=j\rangle,$$

where $m'=j, j-1, j-2, \ldots$ in terms of Q and the appropriate Clebsch-Gordan coefficientes.

P.10. P.17.9, Merzbacher:
A system that is invariant under rotation is perturbed by a quadrupole interaction

$$V = \sum_{q=-2}^{q=+2} C_q T_q^{(2)}$$

where the C_q are constant coefficients and $T_q^{(2)}$ are the components of an irreducible spherical tensor operator, defined by one of its components:

$$T_2^{(2)} = (J_x + iJ_y)^2 \, .$$

a) Deduce the conditions for the coefficients C_q if V is to be Hermitian.

b) Consider the effect of the quadrupole perturbation on the manifold of a degenerate energy eigenstate of the unperturbed system with angular momentum quantum number j, neglecting all other unperturbed energy eigenstates. What is the effect of the perturbation on the manifold of an unperturbed $j = 1/2$ state? Note that it is necessary to determine the matrix elements $\langle j = 1/2 \, m|V|j = 1/2 \, m'\rangle$.

c) If $C_{\pm 2} = C_0$ and $C_{\pm 1} = 0$, calculate the perturbed energies for a $j = 1$ state in terms of C_0.

Problemas adicionais:

11. P.13.15, Messiah:
Let a_r, $a_r^\dagger$, with $r = 1, 2$, be the annihilation and creation operators of a two-dimensional, isotropic, harmonic oscillator:

$$[a_r, a_s] = [a_r^\dagger, a_s^\dagger] = 0, \qquad [a_r, a_s^\dagger] = \delta_{r,s}.$$

We write

$$S = \frac{1}{2}\left(a_1^\dagger a_1 + a_2^\dagger a_2\right), \qquad J_1 = \frac{1}{2}\left(a_2^\dagger a_1 + a_1^\dagger a_2\right),$$

$$J_2 = \frac{i}{2}\left(a_2^\dagger a_1 - a_1^\dagger a_2\right), \qquad J_3 = \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right).$$

J_1, J_2, and J_3 may be considered as the Cartesian coordinates of a certain vector operator $\mathbf{J}$.

a) Show that the components of $\mathbf{J}$ verify the commutation relations $\mathbf{J} \times \mathbf{J} = i\mathbf{J}$ characteristic of an angular momentum, and that one has $J^2 = S(S+1)$, and therefore, $[S, \mathbf{J}] = 0$.

b) $\mathbf{J}$ will henceforth be considered to be the angular momentum of the system, and we denote the eigenvalues of J^2 and J_3 by $j(j+1)$ and m respectively. Show that J^2 and J_3 form a complete set of commuting observables, and that j may take all integral and half-integral values, i.e., $j = 0, 1/2, 1, 3/2, 2, \ldots$. Show that the vectors

$$[(j+m)!(j-m)!]^{-1/2}(a_1^\dagger)^{j+m}(a_2^\dagger)^{j-m}|0\rangle$$

form the basis of a standard representation J^2, J_3.

c) Show that $a_1^\dagger$ and $a_2^\dagger$ are respectively the $+1/2$ and $-1/2$ components of an irreducible tensor operator of order $1/2$, and that it follows that the expressions $Ra_r^\dagger R^{-1}$ $(r = 1, 2)$, where R denotes the rotation operator

$$R = \exp(-i\alpha J_3)\exp(-i\beta J_3)\exp(-i\gamma J_3),$$

are linear combinations of $a_1^\dagger$ and $a_2^\dagger$. Determine the coefficients.

12. P.13.19, Messiah: Let K_u denote the component of a vector operator $\mathbf{K}$ in a given direction, J_u the component of the total angular momentum $\mathbf{J}$ in the same direction, and $|\tau\, J\, a\rangle$, $|\tau\, J\, b\rangle$ be two ket vectors belonging to the same subspace $\mathcal{E}(\tau\, J)$ (definition of Sec. 13.16). Show that:

$$\langle \tau\, J\, a|K_u|\tau\, J\, b\rangle = \langle \tau\, J\, a|J_u|\tau\, J\, b\rangle \frac{\langle \mathbf{J}\cdot\mathbf{K}\rangle}{J(J+1)},$$

where $\langle \mathbf{J}\cdot\mathbf{K}\rangle$ denotes the average value of the scalar operator $\mathbf{J}\cdot\mathbf{K}$ in this subspace:

$$\langle \mathbf{J}\cdot\mathbf{K}\rangle = \langle \tau\, J\, a|\mathbf{J}\cdot\mathbf{K}|\tau\, J\, b\rangle,$$

i.e. the elements of the matrices of $\mathbf{K}$ in $\mathcal{E}(\tau\, J)$ are the same as those of its "projection" $\mathbf{J}(\mathbf{J}\cdot\mathbf{K})/J(J+1)$.

13. Ex.17.13, Merzbacher:
Show that a symmetry exists among the three quantum numbers J_1, J_2, and j, and that in addition to (17.57), they satisfy the equivalent relations

$$|j - J_2| \leq j_1 \leq j + J_2 \qquad \text{and} \qquad |j - J_1| \leq j_2 \leq j + J_1.$$

14. P.10.2, Cohen-Tannoudji: Landé factor.

15. P.10.5, Cohen-Tannoudji: Total angular momentum of 3 spin $S = 1/2$ particles.

16. P.5.3, Desai:
Obtain the expectation values of S_x, S_y, and S_z for the case of a spin $1/2$ particle with the spin pointed in the direction of a vector with azimuthal β and polar α angles.

17. P.5.6, Desai:
If a Hamiltonian is given by $\hbar\sigma \cdot \hat{n}$, where $\hat{n} = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$, determine the time evolution operator as a 2×2 matrix. If a state at $t = 0$ is given by

$$|\phi(0)\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

then obtain $|\phi(t)\rangle$.

18. P.28.3, Desai:
A system consisting of two spin $1/2$ particles is subjected to a time-dependent perturbation

$$V = \theta(t)\frac{\lambda}{\hbar}\mathbf{S}_1 \cdot \mathbf{S}_2,$$

where λ is a constant. The initial state $|\psi(t=0)\rangle$ of the system is given by $|-\,+\rangle$. Express V in terms of the total spin in triplet and singlet states through the Pauli operator with the total spin $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$. Obtain $|\psi(t)\rangle$ for $t > 0$ and determine the probability that it is in the $|+\,+\rangle$, $|+\,-\rangle$, and $|-\,-\rangle$.

19. P.3.25, Sakurai:
Prove, for any j,

$$\sum_{m=-j}^{j} m^2 \left| d^{(j)}_{m\,m'}(\beta) \right| = \frac{1}{2}j(j+1)\sin^2\beta + \frac{1}{2}m'^2(3\cos^2\beta - 1).$$

Hint: This can be proved in many ways. You may, for instance, examine the rotational properties of J_Z^2 using the spherical (irreducible) tensor language.

20. P.3.31, Sakurai:
Consider a spinless particle bound to a fixed center by a central force potential.

a) Relate, as much as possible, the matrix elements

$$\langle n', l', m'| \mp \frac{1}{\sqrt{2}}(x \pm iy)|n, l, m\rangle \qquad \text{and} \qquad \langle n', l', m'|z|n, l, m\rangle$$

using only the Wigner-Eckart theorem. Make sure to state under what conditions the matrix elements are nonvanishing.

b) Do the same problem using wave functions $\psi(\mathbf{r}) = R_{nl}(r)Y_l^m(\theta, \phi)$.

Useful relations:

$$
\begin{aligned}
|2,2\rangle &= |1\rangle|1\rangle \\
|2,1\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle + |1\rangle|0\rangle\right) \\
|2,0\rangle &= \frac{1}{\sqrt{6}}\left(|-1\rangle|1\rangle + 2|0\rangle|0\rangle + |1\rangle|-1\rangle\right) \\
|2,-1\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle|-1\rangle + |-1\rangle|0\rangle\right) \\
|2,-2\rangle &= |-1\rangle|-1\rangle
\end{aligned}
$$

$$
\begin{aligned}
|1,1\rangle &= \frac{1}{\sqrt{2}}\left(|1\rangle|0\rangle - |0\rangle|1\rangle\right) \\
|1,0\rangle &= \frac{1}{\sqrt{2}}\left(|1\rangle|-1\rangle - |-1\rangle|1\rangle\right) \\
|1,-1\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle|-1\rangle - |-1\rangle|0\rangle\right)
\end{aligned}
$$

$$
|0,0\rangle = \frac{1}{\sqrt{3}}\left(|-1\rangle|1\rangle - |0\rangle|0\rangle + |1\rangle|-1\rangle\right)
$$

$$
\begin{aligned}
|j = l + 1/2\rangle &= +\frac{\sqrt{l+m+1/2}}{\sqrt{2l+1}}|m-1/2,\ 1/2\rangle + \frac{\sqrt{l-m+1/2}}{\sqrt{2l+1}}|m+1/2,\ -1/2\rangle \\
|j = l - 1/2\rangle &= -\frac{\sqrt{l-m+1/2}}{\sqrt{2l+1}}|m-1/2,\ 1/2\rangle + \frac{\sqrt{l+m+1/2}}{\sqrt{2l+1}}|m+1/2,\ -1/2\rangle
\end{aligned}
$$