FI 002 – Mecânica Quântica II – Lista 2

P.01. Constant perturbation slowly turned on: A given system is perturbed by the potential

$$V(t) = V_0 e^{\eta t}, \qquad 0 < \eta \ll 1,$$

where V_0 is a constant. Consider that the system is initially ($t_0 \to -\infty$) in the state $|a\rangle$ and apply first-order perturbation theory to determine the transition probability $W_{ba}(t)$ to a state $|b\rangle$ at the instant t. Determine the transition rate w (transition probability per unit time) and then consider the limit $\eta \to 0$.

P.02. P.19.1, Merzbacher:
Calculate the cross section for the emission of a photoelectron ejected when linearly polarized monochromatic light of frequency ω is incident on a complex atom. Simulate the initial state of the atomic electron by the ground state wave function of an isotropic three-dimensional harmonic oscillator (angular frequency ω_0) and the final state by a plane wave. Obtain the angular distribution as a function of the angle of emission.
Hints: P.5.40, Sakurai and Sec. 19.6, Merzbacher.

03. P.19.2, Merzbacher:
Calculate the total cross section for photoemission from the K shell as a function of the frequency of the incident light and the frequency of the K-shell absorption edge, assuming that $\hbar\omega$ is much larger than the ionization potential but that nevertheless the photon momentum is much less than the momentum of the ejected electron. Use a hydrogenic wave function for the K shell and plane waves for the continuum states.
Hint: Sec. 19.6, Merzbacher.

P.04. Ex.19.2, Merzbacher:
Apply first-order perturbation theory to a forced linear harmonic oscillator [Eq. (14.105)] which is initially ($t_0 \to -\infty$) in the ground state. Determine the probability W_{on} to a transition to an excited state $|n\rangle$ when $t \to +\infty$ and compare with the exact result [Eq. (14.146)]. Calculate the energy transfer to the oscillator exactly and also in perturbation theory. Explain the agreement.
Finally, assume that $Q(t) = \mathcal{E}_0 e^{-\eta|t|}\cos\omega_0 t$ and $P(t) = 0$, determine W_{on} and then consider the limit $\eta \to 0$.

P.05. P.5.30, Sakurai:
Consider a two-level system described by the Hamiltonian H_0 such that $H_0|n\rangle = E_n|n\rangle$, with $n = 1, 2$ and $E_1 < E_2$. There is a time-dependent potential $V(t)$ that connects the two levels as follows:

$$V_{11} = V_{22} = 0, \qquad V_{12} = \gamma e^{i\omega t}, \qquad V_{21} = \gamma e^{-i\omega t},$$

where γ is a real number.

a) Consider that $|\psi(t)\rangle = \sum_n c_n(t) e^{-iE_n t/\hbar}|n\rangle$ and show that the coefficients $c_n(t)$ satisfy coupled differential equations

$$i\hbar\dot{c}_k = \sum_{n=1}^{2} V_{kn}(t)e^{i\omega_{kn}t}c_n, \qquad k = 1, 2.$$

b) At $t = 0$, it is known that only the lower level is populated, that is, $C_1(0) = 1$ and $C_2(0) = 0$. Find $|C_1(t)|^2$ and $|C_2(t)|^2$ for $t > 0$ exactly solving the coupled differential equations determined in item (a).

c) Do the same problem using time-dependent perturbation theory to lowest non-vanishing order. Compare the two approaches for small values of γ. Treat the following two cases separately: (i) ω very different from ω_{21} and (ii) ω close to ω_{21}.

P.06. P.5.37, Sakurai:
Consider a neutron in a magnetic field, fixed at an angle θ with respect to the z-axis, but rotating slowly in the ϕ-direction

$$\mathbf{B}(t) = B\left(\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}\right),$$

where $\phi = \phi(t) = \omega t$, with $\omega \ll 1$, i.e., the tip of the magnetic field traces out a circle on the surface of the sphere at "latitude" $\pi - \theta$.

a) Consider that the Hamiltonian of the system is

$$H(t) = -\vec{\mu}\cdot\mathbf{B} = -\frac{1}{2}\hbar\omega_1\hat{\sigma}\cdot\hat{B}(t)$$

where $\hat{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ with σ_i being a Pauli matrix and determine the instantaneous basis $\chi_+(t)$ and $\chi_-(t)$, i.e, solve the eigenvalue problem

$$H(t)\chi_n(t) = E_n(t)\chi(t), \qquad n = \pm,$$

for fixed t. Assume that $\Psi(0) = \chi_+(0)$ and show that $\Psi(t) = c_+(t)\chi_+(t) + c_-(t)\chi_-(t)$, where

$$c_+(t) = \left(\cos\Omega t + i\frac{\omega_1 + \omega\cos\theta}{2\Omega}\sin\Omega t\right)e^{-i\omega t/2}, \quad c_+(t) = \left(\frac{i\omega}{2\Omega}\sin\Omega t\right)e^{-i\omega t/2},$$

and $\Omega = (\omega^2 + \omega_1^2 + 2\omega\omega_1\cos\theta)^{1/2}/2$ [see Eq.(74.3), lecture notes]. Finally, consider that $\omega \approx 0$ (adiabatic condition) and determine the Berry phase γ_+.

b) Alternative item (a): Explicitly calculate the Berry potential $\mathbf{A}$ for the spin-up state from (5.6.23), take its curl, and determine Berry's phase γ_+. Thus, verify (5.6.42) for this particular example of a curve C.

P.07. P.5.29, Sakurai:
Consider a composite system made up of two spin$-1/2$. For $t < 0$, the Hamiltonian does not depend on spin and can be taken to be zero by suitably adjusting the energy scale. For $t > 0$, the Hamiltonian is given by

$$H = \left(\frac{4\Delta}{\hbar^2}\right) \mathbf{S}_1 \cdot \mathbf{S}_2.$$

Suppose the system is in $|+-\rangle$ for $t \leq 0$. Find, as a function of time, the probability for its being found in each of the following states $|++\rangle$, $|+-\rangle$, $|-+\rangle$, and $|--\rangle$: (a) By solving the problem exactly. (b) By solving the problem assuming the validity of first-order time-dependent perturbation theory with H as a perturbation switched on at $t = 0$. Under what condition does (b) give the correct results?

08. P.5.38, Sakurai:
The ground state of a hydrogen atom ($n = 1, l = 0$) is subjected to a time-dependent potential as follows:

$$V(\mathbf{r}, t) = V_0 \cos(kz - \omega t).$$

Using time-dependent perturbation theory, obtain an expression for the transition rate at which the electron is emitted with momentum $\mathbf{p}$. Show, in particular, how you may compute the angular distribution of the ejected electron (in terms of θ and ϕ defined with respect to the z-axis). Discuss briefly the similarities and the differences between this problem and the (more realistic) photoelectric effect. Follow the assumptions of P.17.2, Messiah for the initial and final states.

09. P.17.1, Messiah:
Let $|u_1\rangle$ and $|u_2\rangle$ be two orthogonal eigenstates corresponding to a doubly degenerate level of the Hamiltonian H_0 of a system. The introduction of a constant perturbation V removes the degeneracy and splits the level into two levels a distance ϵ apart. Suppose that the system is initially in the state $|u_1\rangle$ and that the perturbation V is introduced during a time T. If $W_{1\to2}$ is the probability of finding the system in the state $|u_2\rangle$ after the perturbation has been turned off, show that $W_{1\to2}$ is a periodic function of T with angular frequency $\epsilon/\hbar$ and verify that in the limit when $\epsilon T \ll \hbar$ we obtain the result given by the first-order perturbation theory. What is necessary in order that $W_{1\to2}$ vanish whatever T?

P.10. P.17.2, Messiah:
A hydrogen atom is subject to an oscillating electric field $\vec{\mathcal{E}}(t) = \vec{\mathcal{E}}_0 \cos \omega t$ whose circular frequency ω is greater than its ionization frequency $me^4/2\hbar^3$. If the atom is initially in its ground state, what is the probability per unit time of a transition to an ionized state (suppose that we may use plane waves to represent ionized states)? What is the angular distribution of the electron emitted in this excitation process?
N.B. The process described here is that of the photoelectric effect for which one thus obtains a semi-classical treatment in which the electromagnetic field is not quantized.

11. P.12.1, Baym:
An electron is in the ground state of a three-dimensional isotropic harmonic oscillator (angular frequency ω_0). A uniform electric field, $\vec{\mathcal{E}}(t) = \vec{\mathcal{E}}_0 \cos\omega t$, is applied to the electron. Calculate the rate of ionization of the electron as a function of the frequency ω.

12. P.17.10, Desai:
A harmonic oscillator (angular frequency ω_0) is subjected to an electromagnetic field (laser) such that the interaction Hamiltonian is given by

$$V(q,p,t) = \frac{eE_0}{2}\left(\frac{p}{m\omega}\sin\omega t - q\cos\omega t\right),$$

for $0 < t < \infty$. Determine the probability $W_{0\to 1}(t)$ that at time t the oscillator will make a transition from the ground state to the first excited state. Also obtain

$$w_{0\to 1}(t) = \lim_{t\to\infty}\frac{dW_{0\to 1}(t)}{dt}.$$

P.13. P.17.15, Desai:
Consider a particle bound in a simple harmonic oscillator potential (angular frequency ω). Initially ($t < 0$), it is in the ground state. At $t = 0$, a perturbation of the form

$$V(x,t) = Ax^2 e^{-t/\tau} e^{-i\Omega t}, \qquad \tau > 0,$$

is switched on. Using time-dependent perturbation theory, calculate the probability that after a sufficiently long time ($t \gg \tau$), the system will have made a transition to a given excited state. Consider all final states.

14. P.18.4, Desai:
To take into account spontaneous emission in radiative transitions in atoms one can resort to quantum electrodynamics (QED) and start with designating the combined radiation and atomic states as $|n_k\,\alpha\rangle$, where n_k corresponds to the number of photons and the subscript k stands for the quantum numbers (momentum, polarization, etc.) of the photon, while α indicates the state of the atom. Let the vector potential be written as

$$\mathbf{A}(\mathbf{r},t) = \mathbf{A}_0 \sum_k \left(a_k e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t} + a_k^\dagger e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t}\right)$$

where a_k are operators on the states $|n_k\,\alpha\rangle$ and satisfy the same commutation relations as for the harmonic oscillator case such that $n_k = a_k^\dagger a_k$ and $a_k|n_k\,\alpha\rangle = \sqrt{n_k}|n_k - 1\,\alpha\rangle$. Insert the $\mathbf{A}(\mathbf{r},t)$ in the expression for the matrix element $\langle b|\mathbf{A}\cdot\mathbf{p}|a\rangle$. Show that the absorption term is proportional to n_k while the emission term is proportional to $n_k + 1$. Identify the part of the emission term that corresponds to stimulated emission and the part that corresponds to spontaneous emission.