FI 002 – Mecânica Quântica II – Lista 5

P.01. Properties of the γ matrices.

a) From the properties of the α^i and β matrices,

$$(\alpha^i)^2 = \beta^2 = 1, \qquad \alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij}\hat{1}_{4\times4}, \qquad \alpha^i\beta + \beta\alpha^i = 0,$$

show that

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\hat{1}_{4\times4}.$$

b) Show that the γ^5 matrix can be written as

$$\gamma^5 = \gamma_5 = \frac{i}{4!}\epsilon_{\alpha\beta\mu\nu}\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu$$

and that it obeys the following relations

$$\left(\gamma^5\right)^2 = 1, \qquad \gamma^{5\dagger} = \gamma^5, \qquad \left\{\gamma^\mu, \gamma^5\right\} = 0.$$

Recall that $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$.

c) For a product of an odd number of γ matrices, show that $\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\cdots\gamma^\mu\right) = 0$.

d) For a product of an even number of γ matrices, show that

$$\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\right) = 4g^{\alpha\beta}, \qquad \mathrm{Tr}\left(\sigma^{\alpha\beta}\right) = 0,$$

$$\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) = 4\left(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu}\right).$$

e) Show that the traces involving the γ^5 matrices are given by

$$\mathrm{Tr}\left(\gamma^5\right) = \mathrm{Tr}\left(\gamma^5\gamma^\alpha\right) = \mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\right) = \mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\mu\right) = 0,$$

$$\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) = -4i\epsilon^{\alpha\beta\mu\nu}.$$

P.02. Consider the positive $\psi_r^{(+)}(x)$ and negative $\psi_r^{(-)}(x)$ energy solutions of the free–Dirac equation, where $r = 1, 2$.

a) Determine the components of the current density four–vector $J^\mu(x) = c\bar{\psi}(x)\gamma^\mu\psi(x)$ for the $\psi_1^{(+)}(x)$ and $\psi_2^{(-)}(x)$.

b) Apply the charge conjugation transformation $C = i\gamma^2 K_0$ to $\psi_r^{(-)}(x)$ with $r = 1, 2$ and explicitly determine $\psi_{r,c}^{(-)}(x)$.

P.03. P.A.1.1, Mandl and Shaw:
From $\gamma^{\nu} = \Lambda^{\nu}_{\mu} S \gamma^{\mu} S^{-1}$, where $S = S(\Lambda)$, prove that

$$S^{-1}\gamma^5 S = \gamma^5 \det\Lambda.$$

Hence, show the transformation properties of the bilinear covariants:

$$\begin{array}{ll} \bar{\psi}(x)\psi(x) & \text{is a scalar,} \\ \bar{\psi}(x)\gamma^{\mu}\psi(x) & \text{is a four}-\text{vector,} \\ \bar{\psi}(x)\sigma^{\mu\nu}\psi(x) & \text{is a antisymmetric tensor,} \\ \bar{\psi}(x)\gamma_5\gamma^{\mu}\psi(x) & \text{is a pseudovector, and} \\ \bar{\psi}(x)\gamma_5\psi(x) & \text{is a pseudoscalar.} \end{array}$$

P.04. In the Dirac theory, the current density four-vector is defined as

$$J^{\mu}(x) = c\bar{\psi}(x)\gamma^{\mu}\psi(x).$$

a) Consider the charge conjugation trasformation C and show that

$$J'^{\mu}(x) = c\bar{\psi}_c(x)\gamma^{\mu}\psi_c(x) = J^{\mu}(x),$$

where $\psi_c(x) = C\psi(x) = i\gamma^2\psi^*(x)$.

b) Consider the time-reversal transformation T and show that

$$J'^{\mu}(x') = J'^{\mu}(\mathbf{r}, -t) = J_{\mu}(\mathbf{r}, t) = J_{\mu}(x),$$

where $J'^{\mu}(x') = c\bar{\psi}'(x')\gamma^{\mu}\psi'(x')$ and

$$\psi'(x') = \psi'(\mathbf{r}, t') = \psi'(\mathbf{r}, -t) = i\gamma^3\gamma^1\psi^*(\mathbf{r}, t) = i\gamma^3\gamma^1\psi^*(x).$$

P.05. P.24.7, Merzbacher:
Consider a neutral spin one-half Dirac particle with mass m and with an intrinsic magnetic moment in the presence of a uniform constant magnetic field along the z-axis. The Hamiltonian of the particle reads

$$H = -i\hbar c\hat{\alpha} \cdot \nabla + \beta mc^2 + \lambda B \beta \Sigma_z,$$

where the coefficient λ is a constant, proportional to the gyromagnetic ratio. Determine the commutators

$$[H, p_i], \quad [H, L_i], \quad [H, \Sigma_i], \quad [H, J_i]$$

where

$$\mathbf{J} = \mathbf{L} + \frac{\hbar}{2}\hat{\Sigma}, \quad \hat{\Sigma} = \left(\sigma^{23}, \sigma^{31}, \sigma^{12}\right), \qquad \text{and} \qquad \sigma^{ij} = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

with $i, j, k = 1, 2, 3$ in cyclic order, and then determine the important constants of the motion. Derive the energy eigenvalues. Show that the orbital and the spin motions are coupled in the relativistic theory but decoupled in a nonrelativistic limit.

P.06. P.6.15, Schwabl:
The spin projection operator is defined as

$$\Sigma(n) = \frac{1}{2}\left(1 + \gamma_5 \not{n}\right),$$

where $n_\mu n^\mu = -1$ and $n_\mu p^\mu = 0$.

a) Show that

$$\Sigma^2(n) = \Sigma(n), \qquad [\Lambda_\pm(p), \Sigma(n)] = 0, \qquad \mathrm{Tr}\,[\Lambda_\pm(k)\Sigma(\pm n)] = 1,$$

$$\Lambda_+(p)\Sigma(n) + \Lambda_-(p)\Sigma(n) + \Lambda_+(p)\Sigma(-n) + \Lambda_-(p)\Sigma(-n) = 1,$$

where $\Lambda_\pm(p)$ are the energy projection operators.

b) Consider the particle rest frame and $n'^\mu = (0, 0, 0, 1)$. Determine $\Sigma(n')$ and $\Sigma(-n')$ in matrix form and calculate $\Sigma(\pm n')u_r(0)$ and $\Sigma(\pm n')v_r(0)$, where $u_r(0)$ and $v_r(0)$ with $r = 1, 2$ are the solutions of the free–Dirac equation.

c) Consider that the particle is moving along the z-axis and determine $\Sigma(n)$ in matrix form, where $n_\mu = \Lambda^\nu{}_\mu n'_\nu$. Consider the solutions of the free-Dirac equation given by Eq. (286.2) from the lecture notes and calculate $\Sigma(n)u_1(\mathbf{p})$ and $\Sigma(n)u_2(\mathbf{p})$.

P.07. The helicity operator is defined as

$$\sigma_{\mathbf{p}} = \frac{\mathbf{\Sigma}\cdot\mathbf{p}}{|\mathbf{p}|} = \mathbf{\Sigma}\cdot\hat{p}, \qquad \text{where} \quad \mathbf{\Sigma} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \quad \text{and} \quad \sigma = (\sigma^1, \sigma^2, \sigma^3).$$

a) Show that $\sigma_{\mathbf{p}}^2 = 1$ and determine the eigenvalues of the helicity operator.

b) Show that $\sigma_{\mathbf{p}}$ commutes with the Dirac Hamiltonian H_D. Therefore, it is possible to find spinors $u_r(\mathbf{p})$ and $v_r(\mathbf{p})$, the positive and negative energy solutions of the free-particle Dirac equation respectively, with a well-defined helicity. We then assume that

$$\sigma_{\mathbf{p}} u_r(\mathbf{p}) = (-1)^{r+1} u_r(\mathbf{p}), \qquad \sigma_{\mathbf{p}} v_r(\mathbf{p}) = (-1)^r v_r(\mathbf{p}), \qquad r = 1, 2.$$

c) The helicity projection operators are defined as

$$\Pi^\pm(\mathbf{p}) = \frac{1}{2}\left(1 \pm \sigma_{\mathbf{p}}\right).$$

Show that

$$\left[\Pi^\pm(\mathbf{p})\right]^2 = \Pi^\pm(\mathbf{p}), \quad \Pi^\pm(\mathbf{p})\Pi^\mp(\mathbf{p}) = 0, \quad \Pi^+(\mathbf{p}) + \Pi^-(\mathbf{p}) = 1,$$

and

$$\left[\Lambda_\pm(\mathbf{p}), \Pi^\pm(\mathbf{p})\right] = 0,$$

where $\Lambda_\pm(p)$ are the energy projection operators.

d) Consider the solutions of the free-Dirac equation shown in item (b) and determine $\Pi^{\pm}(\mathbf{p})u_r(\mathbf{p})$ and $\Pi^{\pm}(\mathbf{p})v_r(\mathbf{p})$.

e) Consider the solutions of the free-Dirac equation given by Eq. (286.2) from the lecture notes with $\mathbf{p} = (0, 0, p)$ and determine $\sigma_{\mathbf{p}} u_r(\mathbf{p})$ and $\sigma_{\mathbf{p}} v_r(\mathbf{p})$, with $r = 1, 2$.

f) Determine the solutions of the free–Dirac equation such that the spinors $u_r(\mathbf{p})$ and $v_r(\mathbf{p})$ with $r = 1, 2$ are also eigenspinors of the helicity operator $\sigma_{\mathbf{p}}$.
Hint: See Sec. 11.5, Schwabl.

P.08. P.A.1.2, Mandl and Shaw:
For any two positive energy solutions $u_r(\mathbf{p})$ and $u_s(\mathbf{k})$ of the free-particle Dirac equation, show that

$$\bar{u}_s(\mathbf{k}) \left[\not{a}(\not{p} - mc) + (\not{k} - mc)\not{a} \right] u_r(\mathbf{p}) = 0,$$

where a_μ is an arbitrary four-vector. Then, prove the Gordon's identity

$$2mc\bar{u}_s(\mathbf{k})\gamma^\mu u_r(\mathbf{p}) = \bar{u}_s(\mathbf{k}) \left[(k^\mu + p^\mu) + i\sigma^{\mu\nu}(k_\nu - p_\nu) \right] u_r(\mathbf{p}).$$

P.09. P.11.2, Schwabl:
In the Majorana representation, the γ matrices are purely imaginary,

$$(\gamma_M^\mu)^* = -\gamma_M^\mu, \qquad \mu = 0, 1, 2, 3.$$

In this case, the Dirac equation is a real equation and therefore, if ψ is a solution of the Dirac equation, $\psi_c = \psi^*$ is also a solution. A particular Majorana representation is obtained from the Dirac-Pauli representation by the unitary transformation U,

$$\gamma_M^\mu = U\gamma^\mu U^\dagger, \qquad U = \frac{1}{\sqrt{2}}\gamma^0(1 + \gamma^2).$$

a) Show that $U^\dagger = U^{-1}$ and that

$$\gamma_M^0 = \gamma^0\gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \qquad \gamma_M^1 = \gamma^2\gamma^1 = i\sigma^{12} = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix},$$

$$\gamma_M^2 = -\gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \qquad \gamma_M^3 = \gamma^2\gamma^3 = -i\sigma^{23} = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix},$$

$$\gamma_M^5 = -i\gamma^0\gamma^1\gamma^3 = \gamma^0\sigma^{31} = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}.$$

b) In a Majorana representation, the charge conjugation has the form $\psi_M^C = \psi_M^*$, apart from an arbitrary phase factor. Apply the transformation U to $\psi^C(x) = i\gamma^2\psi^*(x)$ and show that $\psi_M^C(x) = -i\psi_M^*(x)$.

10. Consider a particular representation for the γ matrices that is derived from the Dirac–Pauli matrices γ^μ via the unitary transformation U,

$$\gamma^\mu_w = U\gamma^\mu U, \qquad \text{where} \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

with $1 = \hat{1}_{2\times 2}$ being the identity matrix. Determine the γ^μ_w matrices and then write the Dirac equation in this new representation. In particular, assume that the four–component spinor $\psi'(x) = U\psi(x)$ can be written in terms of two-component spinors

$$\psi'(x) = U\psi(x) = \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix}.$$

Discuss the case of a massless particle $m = 0$.

11. The energy projection operators are defined as

$$\Lambda^\pm(p) = \frac{\pm \not{p} + mc}{2mc}.$$

a) Show that

$$[\Lambda_\pm(p)]^2 = \Lambda_\pm(p), \quad \Lambda_+(p)\Lambda_-(p) = 0, \quad \mathrm{Tr}\Lambda_\pm(p) = 2, \quad \Lambda_+(p) + \Lambda_-(p) = 1.$$

b) Show that the projection operators can be written as

$$\Lambda^+_{\alpha\beta}(p) = \sum_{r=1}^{2} u_{r\alpha}(\mathbf{p})\overline{u}_{r\beta}(\mathbf{p}), \qquad \Lambda^-_{\alpha\beta}(p) = -\sum_{r=1}^{2} v_{r\alpha}(\mathbf{p})\overline{v}_{r\beta}(\mathbf{p}).$$

where $u_r(\mathbf{p})$ and $v_r(\mathbf{p})$ with $r = 1, 2$ are the solutions of the free–Dirac equation.

c) Show that the completeness relation holds

$$\sum_{r=1}^{2} [u_{r\alpha}(\mathbf{p})\overline{u}_{r\beta}(\mathbf{p}) - v_{r\alpha}(\mathbf{p})\overline{v}_{r\beta}(\mathbf{p})] = \delta_{\alpha,\beta}.$$

d) Show that the projection operators can be written as

$$\Lambda^\pm(p) = \frac{1}{2}\left[1 \pm \frac{c\hat{\alpha}\cdot\mathbf{p} + mc^2}{E}\right].$$

12. P.7.1, Schwabl:
Consider a spin one-half Dirac particle with mass m and charge q under a central potential $\phi(r)$ with $r = |\mathbf{r}|$. The Hamiltonian of the particle reads

$$H = -i\hbar c\hat{\alpha}\cdot\nabla + \beta mc^2 + q\phi(r).$$

Show that $[H, J_i] = 0$, where J_i is a component of the total angular momentum $\mathbf{J} = \mathbf{L} + (\hbar/2)\hat{\Sigma}$.

13. P.23.1, Baym:
Suppose that an electron of momentum $\mathbf{p}$ incident from the left strikes a (one-dimensional) potential barrier

$$q\Phi(x) = V_0, \ \ x > 0, \quad \text{and} \quad q\Phi(x) = 0, \ \ x < 0,$$

Calculate the reflection from the barrier for $V_0 < 2mc^2$ and for $V_0 > 2mc^2$ [Klein paradox]. Interpret the results for $V_0 > 2mc^2$ in terms of the Dirac hole theory.
Hint: See Sec. 10.1.4, Schwabl.

14. P.24.2, Merzbacher:
If a field theory of massless spin one-half particles (neutrinos) is developed, so that the β matrix is absent, show that the conditions

$$(\alpha^i)^2 = \beta^2 = 1, \qquad \alpha^i\alpha^j + \alpha^j\alpha^i = 0 \quad i \neq y, \qquad \alpha^i\beta + \beta\alpha^i = 0.$$

are solved by 2×2 Pauli matrices, $\hat{\alpha} = \pm\hat{\sigma}$. Work out the details of the resulting *two-component* theory with particular attention to the helicity properties. Is this theory invariant under spatial reflection?

15. P.24.8, Merzbacher:
If a Dirac electron is moving in a uniform constant magnetic field pointing along the z-axis, determine the energy eigenvalues and eigenspinors.
Hint: See Problem 6.5, Schwabl.

16. P.10.1, Jackson:
The Dirac current density is $\mathbf{J} = ec(\psi^\dagger\hat{\alpha}\psi)$.

a) Show that this expression can be transformed into

$$\mathbf{J} = \mathbf{J}_s + c\nabla \times \mathbf{M} + \partial_t \Pi,$$

where $\mathbf{J}_s$ is a relativistic generalization of the Schrödinger current, $\mathbf{M}$ is the magnetization, and Π is the electric polarization:

$$\mathbf{J}_s = \frac{e\hbar}{2mi}\left[\psi^\dagger\beta\nabla\psi - (\nabla\psi^\dagger)\beta\psi\right] - \frac{e^2}{mc}\mathbf{A}\psi^+\beta\psi,$$

$$\mathbf{M} = \frac{e\hbar}{2mc}\psi^+\beta\hat{\sigma}_D\psi, \qquad \Pi = \frac{e\hbar}{2imc}\psi^+\beta\hat{\alpha}\psi, \qquad \hat{\sigma}_D = \begin{pmatrix} \hat{\sigma} & 0 \\ 0 & \hat{\sigma} \end{pmatrix}$$

b) Show that the nonrelativistic (Pauli) approximation to $\mathbf{J}_s$ and $\mathbf{M}$ is equivalent to the forms in part (a) with β set equal to unity and ψ interpreted as a Pauli spinor.

c) Find the nonrelativistic approximation to the polarization Π and show that it is of order v/c times the magnetization $\mathbf{M}$.