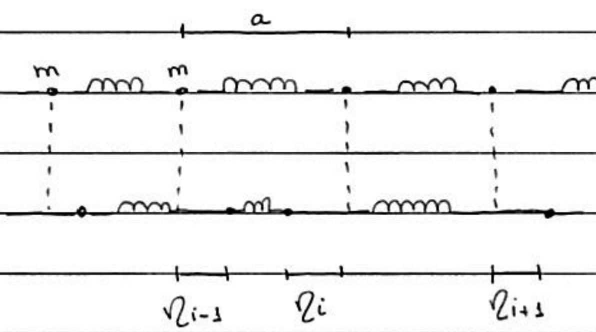
Quantização campos relativísticos.

Refs.: Caps. 11-14, Bjorken and Drell, Relativistic Q.F.
Caps. 2-5, Mandl and Shaw
Caps. 12-14, Schwabl

· inicial: revisão formalismo lagrangiano p/ meios contínuos
(p/ detalhes, veja Sec. 13.2, Goldstein, 3ª ed.)

considerar: sistema 1-D, N partículas massas m conectadas por molas, cte mola k; distância entre partículas equilíbrio = a.



: configuração equilíbrio

: " fora "

se $\eta_i = \eta_i(t)$: deslocamento partícula i w.r.t. posição equilíbrio

↳ $$L = T - U = \frac{1}{2}\sum_{i=1}^{N} m\dot{\eta}_i^2 - k(\eta_{i+1}-\eta_i)^2 \quad (309.1)$$: lagrangiana sistema discreto

Eq. (309.1) pode ser escrita como

$$L = \frac{1}{2}\sum_{i=1}^{N} a\left(\frac{m}{a}\dot{\eta}_i^2 - ka\frac{1}{a^2}(\eta_{i+1}-\eta_i)^2\right) = \sum_{i=1}^{N} aL_i$$

considerando o limite $N \to +\infty$, $a \to 0$, verifica-se que

$$L = \sum_{i=1}^{N} aL_i \to \int dx\, \mathcal{L}$$: lagrangiana sistema contínuo

onde

$$\mathcal{L} = \frac{1}{2}\mu\left(\frac{\partial\eta}{\partial t}\right)^2 - \frac{1}{2}Y\left(\frac{\partial\eta}{\partial x}\right)^2 \text{ : densidade de lagrangiana} \quad (310.1)$$

nesse caso: $\eta_i(t) \to \eta(x,t)$
$m/a \to \mu$: densidade de massa
Y : módulo de Young

notar: $\mathcal{L} = \mathcal{L}(\partial_x\eta, \partial_t\eta)$!

caso geral: considerar sistema 1-D (discreto) descrito pelas coordenadas generalizadas $q_1, q_2, \ldots, q_N$;
a generalização p/ caso $N \to +\infty$ pode ser obtida via as identificações:

$i \to x$ ou $ai \to x$
$q_i(t) \to \phi(x,t)$: campo escalar
$\dot{q}_i(t) \to \partial_t\phi(x,t)$

$$L = \sum_{i=1}^{N} L_i \to \int dx\, \mathcal{L} \quad (310.2)$$

onde (caso geral) $\mathcal{L} = \mathcal{L}(\phi, \partial_x\phi, \partial_t\phi, x, t)$: densidade de lagrangiana;
além disso, a ação S assume a forma

$$S = \int_{-\infty}^{+\infty} dx \int_1^2 dt\, \mathcal{L} \quad (310.3)$$

e o princípio de Hamilton p/ meios contínuos é dado por

$$\delta S = \delta\int dx\,dt\,\mathcal{L} = 0 \quad (310.4)$$

próxima etapa: determinar eq. de Lagrange p/ campo escalar $\phi = \phi(x,t)$ a partir (310.4)

seja $\phi_0(x,t)$: configuração campo escalar que corresponde extremo ação S ;

considerar variação: $\phi(x,t) = \phi_0(x,t) + \alpha \xi(x,t)$
tal que $\xi(x \to \pm\infty, t) = \xi(x, t=t_1) = \xi(x, t=t_2) = 0$

↳ $\mathcal{L} = \mathcal{L}(\phi_0 + \alpha\xi, \partial_x\phi_0 + \alpha\partial_x\xi, \partial_t\phi_0 + \alpha\partial_t\xi, x, t)$

↳ ação $S = S[\alpha]$

nesse caso, δS pode ser determinado via

$$\frac{dS}{d\alpha} = \int dx\,dt \left(\frac{\partial\mathcal{L}}{\partial\phi}\underbrace{\frac{\partial\phi}{\partial\alpha}}_{\xi} + \frac{\partial\mathcal{L}}{\partial(\partial_x\phi)}\underbrace{\frac{\partial}{\partial\alpha}(\partial_x\phi)}_{\partial_x\xi} + \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\underbrace{\frac{\partial}{\partial\alpha}(\partial_t\phi)}_{\partial_t\xi} \right)$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_x\phi)}\partial_x\xi = -\partial_x\left(\frac{\partial\mathcal{L}}{\partial(\partial_x\phi)}\right)\xi + \underbrace{\frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial(\partial_x\phi)}\xi\right)}_{=0,\ \text{pois}\ \xi(x\to\pm\infty,t)=0}$$

procedendo de modo análogo p/ o terceiro termo, temos que

$$\frac{dS}{d\alpha} = \int dx\,dt \left(\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial(\partial_x\phi)}\right) - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\right)\right)\xi(x,t)$$

como $\xi = \xi(x,t)$ é arbitrário, a condição $dS/d\alpha = 0$

↳ $$\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial(\partial_x\phi)}\right) - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\right) = 0 : \qquad (311.1)$$

: Eq. de Lagrange

como $\frac{\partial}{\partial x}$ e $\frac{\partial}{\partial t}$ $\to$ $\frac{\partial}{\partial x^\mu} = \partial_\mu$: Eq. (243.3), temos que

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \qquad (312.1)$$

notar: Eqs. (310.3) e (312.1) podem ser generalizadas p/ caso sistema descrito por campos

$\phi_r(\vec{r},t)$, $r = 1, 2, \ldots, n$

$$S = \int d^3r\, dt\, \mathcal{L} = \int d^4x\, \mathcal{L}$$

onde $\mathcal{L} = \mathcal{L}(\phi_r, \partial_\mu \phi_r, x^\mu)$

e $$\frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \right) = 0 \qquad (312.2)$$

ou

$$\frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{r,\mu}} \right) = 0 \qquad r = 1, 2, \ldots, n \quad \mu = 0, 1, 2, 3$$

definição: $$\frac{\partial \phi_r}{\partial x^\mu} = \partial_\mu \phi_r = \phi_{r,\mu} \qquad (312.3)$$

$$\frac{\partial \phi_r}{\partial x_\mu} = \partial^\mu \phi_r = \phi_r{}^{,\mu}$$

$\hookrightarrow$

· sobre o formalismo hamiltoniano p/ meios contínuos
(p/ detalhes, veja Secs. 13.4, Goldstein, 3ª ed.)

considerar: sistema discreto 1-D, N-partículas massas m
⊕ molas, pg. 309

como p/ 1 partícula: $p_i = \frac{\partial L}{\partial \dot{q}_i}$: momento canonicamente conjugado

↳ $\pi_i = \frac{\partial L}{\partial \dot{\eta}_i} = a\frac{\partial L_i}{\partial \dot{\eta}_i}$: momento canonicamente conjugado à $\eta_i = \eta_i(t)$ (313.1)

como $H = \dot{q}_i p_i - L$

↳ $H = \dot{\eta}_i \pi_i - L = a\frac{\partial L_i}{\partial \dot{\eta}_i}\dot{\eta}_i - a\sum_i L_i =$

$$= \sum_i a\left(\dot{\eta}_i \frac{\partial L_i}{\partial \dot{\eta}_i} - L_i\right) \xrightarrow[N\to+\infty]{a\to 0} H = \int dx \underbrace{\left(\dot{\eta}\frac{\partial \mathcal{L}}{\partial \dot{\eta}} - \mathcal{L}\right)}_{=\mathcal{H}}$$

temos que,

$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\eta}}$: densidade momento canonicamente conjugado à $\eta = \eta(x,t)$

e (313.2)

$\mathcal{H} = \pi\dot{\eta} - \mathcal{L}$: densidade de hamiltoniana

caso geral: sistema descrito por campos $\phi_r = \phi_r(\vec{x},t)$:

$\pi_r = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r}$ e $\mathcal{H} = \pi_r \dot{\phi}_r - \mathcal{L}$ (313.3)

· quantização canônica,

Lembrar: procedimento quantização canônica p/ 1 partícula:

se descrição do sistema clássico correspondente através 2N variáveis independentes (formalismo hamiltoniano)

$q_1, q_2, \ldots, q_N$: coordenadas espaciais (cartesianas) (313.4)

$p_1, p_2, \ldots, p_N$: momentos canonicamente conjugados e, em princípio, ∄ variável dinamica $F = F(q_i, p_j)$

↳ p/ descrição sistema quântico: introdução 2N observáveis correspondentes à (313.4):

$q_1, q_2, \ldots, q_N$ e $p_1, p_2, \ldots, p_N$ tais que:

$$[q_i, q_j] = [p_i, p_j] = 0 \quad e \quad [q_i, p_j] = i\hbar\delta_{ij} \tag{314.1}$$

notar: dados observáveis $A = A(p,q)$ e $B = B(p,q)$, onde
$q: q_1, q_2, \ldots, q_N$ e $p: p_1, p_2, \ldots, p_N$
↳ relações de comutação fundamentais (314.1)
↳ determinação álgebra $[A, B]$!

· similar teoria clássica campos → teoria quântica de campos:

se sistema descrito por

$\phi_r = \phi_r(\vec{x}, t)$
e $\pi_r = \pi_r(\vec{x}, t)$: campos clássicos → $\phi_r(\vec{x}, t)$ e $\pi_r(\vec{x}, t)$: ops. hermiticos tais que (postulado):

$$[\phi_r(\vec{x}, t) ; \phi_s(\vec{x}', t)] = [\pi_r(\vec{x}, t) ; \pi_s(\vec{x}', t)] = 0$$

$$e \quad [\phi_r(\vec{x}, t) ; \pi_s(\vec{x}', t)] = i\hbar\delta_{rs}\delta(\vec{x} - \vec{x}') \tag{314.2}$$

notar Eq. (314.2): relações de comutação p/ instante t fixo!

· Simetrias e leis de conservação,

formalismo lagrangiano: interessante p/ determinação de ctes de movimento em teoria clássica de campos;

verifica-se que

se $\mathcal{L}$: invariante sob uma transformação contínua de simetria $\longrightarrow$ $\exists$ teorema de conservação (eq. de continuidade) e correspondente cte de movimento:

: teorema de Noether (315.1)

próxima etapa: Eq. (315.1) em detalhes;

vamos considerar 2 procedimentos;

(1) hipótese: $\mathcal{L} = \mathcal{L}(\phi_r, \phi_{r,\mu})$, i.e.,

$\mathcal{L}$ não depende explicitamente da posição e do tempo

$\hookrightarrow$ sistema invariante sob translações espacial e temporal (veja abaixo) $\rightarrow$ momento e energia conservados

notar:

$$\frac{\partial \mathcal{L}}{\partial x^\nu} = \partial_\nu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_r} \partial_\nu \phi_r + \frac{\partial \mathcal{L}}{\partial \phi_{r,\mu}} \partial_\nu (\partial_\mu \phi_r) =$$

NOTAR!

Eq de Lagrange (312.2)

$$= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{r,\mu}} \right) \partial_\nu \phi_r + \frac{\partial \mathcal{L}}{\partial \phi_{r,\mu}} \partial_\mu (\partial_\nu \phi_r)$$

$$= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{r,\mu}} \partial_\nu \phi_r \right) = \partial_\nu \mathcal{L} = \delta_\nu{}^\mu \partial_\mu \mathcal{L}$$

$$\hookrightarrow \partial_\mu \Big(\underbrace{\frac{\partial \mathcal{L}}{\partial \phi_{n,\mu}} \partial_\nu \phi_n - \delta_\nu{}^\mu \mathcal{L}}_{\equiv T^\mu{}_\nu} \Big) = 0 \qquad (316.1)$$

como $T^{\alpha\beta} = g^{\beta\nu} T^\alpha{}_\nu = g^{\beta\nu} \frac{\partial \mathcal{L}}{\partial \phi_{n,\alpha}} \partial_\nu \phi_n - g^{\beta\nu} \delta_\nu{}^\mu \mathcal{L}$

$$\hookrightarrow T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \phi_{n,\mu}} \partial^\nu \phi_n - g^{\mu\nu} \mathcal{L}$$: tensor de tensão energia-momento (316.2)

notar: Eq. (316.1) → $T^{\mu\nu}$ satisfaz uma lei de conservação, pois

$$\partial_\mu T^{\mu\nu} = 0 \qquad (316.3)$$

ou, em forma integral,

Teorema do divergente

$$\int_V d^3r\, \partial_0 T^{0\nu} = \frac{1}{c}\frac{d}{dt}\int_V d^3r\, T^{0\nu} = -\int_V d^3r\, \partial_k T^{k\nu} = -\oint_S T^{k\nu}(n_k ds)$$

$$\hookrightarrow \frac{1}{c}\frac{d}{dt}\int_V d^3r\, T^{0\nu} = -\oint_S T^{k\nu}(n_k ds) \qquad (316.4)$$

volume definido pela superfície S !

Definição: $cP^\nu = \int_V d^3r\, T^{0\nu}$: 4-vetor energia-momento dos campos (316.5)

p/ $V \to +\infty$ e campos $\phi_n(\vec{r},t) \xrightarrow[|\vec{r}| \to +\infty]{} 0$,

Eq. (316.4) → $\frac{dP^\nu}{dt} = 0$ ou P^ν : cte de movimento (316.6) notar: $T^{00} = \frac{\partial \mathcal{L}}{\partial \phi_{n,0}} \partial^0 \phi_n - g^{00}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_n} \dot{\phi}_n - \mathcal{L} = \mathcal{H}$

$$T^{0k} = \frac{\partial \mathcal{L}}{\partial \phi_{n,0}} \partial^k \phi_n - g^{0k}\mathcal{L} = c \frac{\partial \mathcal{L}}{\partial \dot{\phi}_n} \frac{\partial \phi_n}{\partial x_k} = c\pi_n \frac{\partial \phi_n}{\partial x_k}$$

↳ $cP^0 = \int_V d^3x\, \mathcal{H} = H$ ou $P^0 = H/c$: hamiltoniano

(317.1)

e $cP^k = c\int_V d^3x\, \pi_n \frac{\partial \phi_n}{\partial x_k}$ ou $\vec{P} = -\int d^3x\, \pi_n \vec{\nabla}\phi_n$:

: momento linear campo

(2) Como uma transf. de simetria contínua = conjunto de transf. infinitesimais

↳ vamos considerar a seguinte transf. infinitesimal

$$x'_\mu = x_\mu + \Delta\omega_{\mu\nu} x^\nu + \delta_\mu = x_\mu + \delta x_\mu$$

δ_μ: parâmetro translação

$\Delta\omega_{\mu\nu}$: " boost / rotação

e (317.2)

$$\phi'_n(x') = \phi_n(x) + \frac{1}{2}\Delta\omega_{\mu\nu} S^{\mu\nu}_{ns}\phi_s(x)$$

onde $x = x^\mu$ e $\phi_n(x)$: REF O

$x' = x'^\mu$ e $\phi'_n(x')$: REF O'

$$\Delta\omega_{\nu\mu} = -\Delta\omega_{\mu\nu} \quad e \quad S^{\nu\mu}_{ns} = -S^{\mu\nu}_{ns}$$

Obs.: p/ detalhes Eq. (317.2), veja pg. 317.1

Definição: $\delta\phi_n(x) = \phi'_n(x) - \phi_n(x)$: argumentos iguais

(317.3)

$\Delta\phi_n(x) = \phi'_n(x') - \phi_n(x)$: variação total

/ /

· sobre Eq. (317.2),

Lembrar: $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$: Eq. (266.2)

e $\Lambda^{\nu}{}_{\mu} = g^{\nu}{}_{\mu} + \Delta\omega^{\nu}{}_{\mu} = \delta^{\nu}{}_{\mu} + \Delta\omega^{\nu}{}_{\mu}$

↳ $x'_{\mu} = g_{\mu\alpha} x'^{\alpha} = g_{\mu\alpha} \Lambda^{\alpha}{}_{\beta} x^{\beta}$

$= g_{\mu\alpha} g^{\alpha}{}_{\beta} x^{\beta} + g_{\mu\alpha} \Delta\omega^{\alpha}{}_{\beta} x^{\beta}$

$= g_{\mu\beta} x^{\beta} + \Delta\omega_{\mu\beta} x^{\beta} = x_{\mu} + \Delta\omega_{\mu\nu} x^{\nu}$

Lembrar: $\hat{S} = \hat{I}_{4\times 4} - \frac{i}{4} \sigma_{\mu\nu} \Delta\omega^{\mu\nu}$: Eq. (274.2)

: matriz transformação p/ spinores

c/ $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$

↳ caso geral: $\phi'_{n}(x') = \phi_{n}(x) + \frac{1}{2} \Delta\omega_{\mu\nu} S^{\mu\nu}_{ns} \phi_{s}(x)$ (317.3)

comparando Eqs. (274.2) e (317.3):

$\frac{1}{2} S^{\mu\nu}_{ns} = -\frac{i}{4} \sigma^{\mu\nu}_{ns}$

↳ elemento matriz $\sigma^{\mu\nu}$.

notar: $\Delta\phi_r(x) = \left(\phi'_r(x') - \phi_r(x')\right) + \left(\phi_r(x') - \phi_r(x)\right)$

$$= \delta\phi_r(x') + \frac{\partial\phi_r}{\partial x_\nu}\delta x_\nu + O(\delta^2)$$

$$= \delta\phi_r(x) + \frac{\partial\phi_r}{\partial x_\nu}\delta x_\nu + O(\delta^2)$$: 1ª ordem em δx_ν : Eq. (317.2)

· invariância sob transformação (317.2)

↳ $\mathcal{L}$ apresenta a mesma forma p/ REFs O e O'

(eqs. de movimento são covariantes):

$$\mathcal{L}\left(\phi'_r(x'), \phi'_{r,\mu}(x')\right) = \mathcal{L}\left(\phi_r(x), \phi_{r,\mu}(x)\right)$$

ou $\mathcal{L}\left(\phi'_r(x'), \phi'_{r,\mu}(x')\right) - \mathcal{L}\left(\phi_r(x), \phi_{r,\mu}(x)\right) = 0$

ou $$\underbrace{\frac{\partial\mathcal{L}}{\partial\phi_r}\delta\phi_r + \frac{\partial\mathcal{L}}{\partial\phi_{r,\mu}}\delta\phi_{r,\mu}}_{\delta\mathcal{L}} + \frac{\partial\mathcal{L}}{\partial x_\mu}\delta x_\mu = 0$$

notar: $\delta\mathcal{L}$ ⊕ Eq. de Lagrange (312.2)

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi_{r,\mu}}\right)\delta\phi_r + \frac{\partial\mathcal{L}}{\partial\phi_{r,\mu}}\underbrace{\delta(\partial_\mu\phi_r)}_{\partial_\mu\delta\phi_r}$$

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi_{r,\mu}}\delta\phi_r\right) \underset{\text{Eq. (317.2)}}{=} \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi_{r,\mu}}\left(\Delta\phi_r(x) - \frac{\partial\phi_r}{\partial x_\nu}\delta x_\nu\right)\right)$$

como: $\frac{\partial\mathcal{L}}{\partial x_\mu}\delta x_\mu = \partial^\mu\mathcal{L}\,\delta x_\mu = \partial_\mu\mathcal{L}\,\delta x^\mu = \partial_\mu(\mathcal{L}\,\delta x^\mu)$

$$= \partial_\mu(\mathcal{L}\,g^{\mu\nu}\delta x_\nu)$$

temos que:

$$\partial_\mu \Big(\underbrace{\frac{\partial \mathcal{L}}{\partial \phi_{n,\mu}} \Delta\phi_n - \Big(\frac{\partial \mathcal{L}}{\partial \phi_{n,\mu}} \partial^\nu \phi_n - g^{\mu\nu}\mathcal{L} \Big)}_{T^{\mu\nu}} \delta x_\nu \Big) = 0$$

↳ $f^\mu = \frac{\partial \mathcal{L}}{\partial \phi_{n,\mu}} \Delta\phi_n - T^{\mu\nu}\delta x_\nu$: 4-vetor densidade de corrente

(319.1)

e $\partial_\mu f^\mu = 0$: eq. de continuidade

Eq. (319.1): teorema de Noether.

· vamos considerar 3 casos particulares Eq. (319.1)

(i) invariância do sistema sob translação;

nesse caso: $\Delta\omega_{\mu\nu} = 0$ e $\delta_\mu \neq 0$

↳ $\delta x_\mu = \delta_\mu$ e $\phi'_n(x') = \phi_n(x) \rightarrow \Delta\phi_n(x) = 0$

Eq. (319.1): $f^\mu = -T^{\mu\nu}\delta_\nu$ e $\partial_\mu f^\mu = -\partial_\mu T^{\mu\nu}\delta_\nu = 0$;

como δ_ν, $\nu = 0,1,2,3$ são arbitrários $\rightarrow \partial_\mu T^{\mu\nu} = 0$: OK c/ Eq. (316.3)

(ii) invariância do sistema sob rotações;

nesse caso: $\Delta\omega_{\mu\nu} \neq 0$ e $\delta_\mu = 0$

$\hookrightarrow \delta x_\mu = \Delta\omega_{\mu\nu} x^\nu$

e $\quad \Delta\phi_n(x) = \phi'_n(x') - \phi_n(x) = \frac{1}{2}\Delta\omega_{\mu\nu} S^{\mu\nu}_{ns}\phi_s(x)$

Eq. (319.1) : $\quad f^\mu = \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\phi_{n,\mu}}\Delta\omega_{\nu\sigma} S^{\nu\sigma}_{ns}\phi_s - T^{\mu\nu}\Delta\omega_{\nu\sigma}x^\sigma \qquad (320.1)$

· Definição : $\quad M^{\mu\nu\sigma} = \frac{\partial\mathcal{L}}{\partial\phi_{n,\mu}} S^{\nu\sigma}_{ns}\phi_s + (x^\nu T^{\mu\sigma} - x^\sigma T^{\mu\nu})$

notar: $\quad T^{\mu\nu}\Delta\omega_{\nu\sigma}x^\sigma = \frac{1}{2}T^{\mu\nu}\Delta\omega_{\nu\sigma}x^\sigma + \underbrace{\frac{1}{2}T^{\mu\nu}\Delta\omega_{\nu\sigma}x^\sigma}_{\frac{1}{2}T^{\mu\sigma}\underbrace{\Delta\omega_{\sigma\nu}}_{=-\Delta\omega_{\nu\sigma}}x^\nu}$

$\hookrightarrow f^\mu = \frac{1}{2}\left(\frac{\partial\mathcal{L}}{\partial\phi_{n,\mu}} S^{\nu\sigma}_{ns}\phi_s + x^\nu T^{\mu\sigma} - x^\sigma T^{\mu\nu}\right)\Delta\omega_{\nu\sigma}$

$= \frac{1}{2} M^{\mu\nu\sigma}\Delta\omega_{\nu\sigma} \qquad (320.2)$

como os 6 parâmetros $\Delta\omega_{\nu\sigma}$ são arbitrários

$\hookrightarrow \partial_\mu M^{\mu\nu\sigma} = 0 \quad$: 6 eqs. de continuidade $\qquad (320.3)$

similar Eq. (316.5), temos 6 ctes de movimento

$c M^{\nu\sigma} = \int_V d^3r \, M^{0\nu\sigma}$

$= \int_V d^3r \, \frac{\partial\mathcal{L}}{\partial\phi_{n,0}} S^{\nu\sigma}_{ns}\phi_s + x^\nu T^{0\sigma} - x^\sigma T^{0\nu}$

$$c\,M^{\nu\sigma} = \int_V d^3x \quad c\,\pi_n S^{\nu\sigma}_{ns} \phi_s + x^\nu T^{0\sigma} - x^\sigma T^{0\nu} \qquad (321.1)$$

em particular:

$$c\,M^{ij} = \int_V d^3x \quad \underbrace{c\,\pi_n S^{ij}_{ns} \phi_s}_{\substack{\text{mom. ang.}\\ \text{intrínseco}}} + \underbrace{\underbrace{x^i T^{0j} - x^j T^{0i}}_{c(x^i p^j - x^j p^i)}}_{\substack{\text{mom. ang.}\\ \text{orbital}}} \quad : \text{momento angular campo} \qquad (321.2)$$

De fato: $\vec{J} = (J^1, J^2, J^3) = (M^{23}, M^{31}, M^{21})$.

(iii) invariância do sistema sob transformação de gauge global.

considerar: $\phi_n(x)$: campo escalar $\in \mathbb{C}$
e transformação campos:

$$\phi_n(x) \to \phi'_n(x) = e^{i\epsilon}\phi_n(x) \simeq (1+i\epsilon)\phi_n(x)$$
$$; \ |\epsilon| \ll 1 \qquad (321.3)$$
$$\phi^+_n(x) \to \phi'^+_n(x) = e^{-i\epsilon}\phi^+_n(x) \simeq (1-i\epsilon)\phi^+_n(x)$$

e $x'_\mu = x_\mu$; notar: $\phi_n(x)$ e $\phi^+_n(x)$: independentes

nesse caso, temos que

$$\delta\phi(n) = \phi'_n(x) - \phi_n(x) = i\epsilon\,\phi_n(x) = \Delta\phi_n(x)$$
$$(321.4)$$
$$\delta\phi^+(n) = \phi'^+_n(x) - \phi^+_n(x) = -i\epsilon\,\phi^+_n(x) = \Delta\phi^+_n(x)$$

$$\delta x_\mu = x'_\mu - x_\mu = 0$$

Eq. (319.1) : $f^{\mu} = \frac{\partial \mathcal{L}}{\partial \phi_{n,\mu}} \Delta\phi_n + \frac{\partial \mathcal{L}}{\partial \phi^{\dagger}_{n,\mu}} \Delta\phi^{\dagger}_n = i\epsilon\left(\frac{\partial \mathcal{L}}{\partial \phi_{n,\mu}} \phi_n - \frac{\partial \mathcal{L}}{\partial \phi^{\dagger}_{n,\mu}} \phi^{\dagger}_{n,\mu}\right)$

(322.1)

em particular,

$$f^0 = i\epsilon c\left(\pi_n(x)\phi_n(x) - \pi^{\dagger}_n(x)\phi^{\dagger}_n(x)\right)$$

dessa forma, temos a cte de movimento:

$$Q = -\frac{i}{\hbar} q \int d^3n \; \pi_n(x)\phi_n(x) - \pi^{\dagger}_n(x)\phi^{\dagger}_n(x) \qquad (322.2)$$

• verifica-se que a cte q ~ carga das partículas

↳ Q : operador carga ;

De fato, Eq. (314.2) $\xrightarrow{\text{VERIFICAR}}$ $[Q, \phi_n(x)] = -q\,\phi_n(x)$

↳ $[Q, \phi^{\dagger}_n(x)] = +q\,\phi^{\dagger}_n(x)$

se $Q|Q'\rangle = Q'|Q'\rangle$

↳ $Q(\phi_n|Q'\rangle) = (Q'-q)\phi_n|Q'\rangle$: ϕ_n : op. destruição partícula carga $+q$

e

$Q(\phi^{\dagger}_n|Q'\rangle) = (Q'+q)\phi_n|Q'\rangle$: $\phi^{\dagger}_n$: " criação " " $+q$

notar: p/ descrição partículas neutras: $\phi_n(x) \in \mathbb{R}$ (322.3)

" " " carregadas: $\phi_n(x) \in \mathbb{C}$

· Campo de Klein-Gordon.

considerar: campo escalar $\phi(x) \in \mathbb{R}$ que satisfaz a eq. de movimento (248.3), Eq. de Klein-Gordon:

$$(\partial^\mu \partial_\mu + \bar{m}^2)\phi(x) = 0 \qquad \bar{m} \equiv \frac{mc}{\hbar} \qquad (323.1)$$

verifica-se a partir Eq. (312.2) que a densidade de lagrangiana cuja eq. de movimento = Eq. (323.1) é dada por (exercício):

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu \phi \, \partial_\mu \phi - \bar{m}^2 \phi^2\right) \qquad (323.2)$$

↳ $\pi(x) = \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dfrac{c\,\partial \mathcal{L}}{\partial_0 \phi} = c\,\partial^0 \phi = \dfrac{1}{c^2}\dot{\phi}(x)$: momento canonicamente conjugado

· procedimento quantização canônica (314.2):

$\phi(x)$: campo clássico $\rightarrow$ $\phi(x) = \phi^\dagger(x)$: op. hermitiano

$\pi(x)$ $\qquad\qquad\qquad\qquad$ $\pi(x) = \pi^\dagger(x)$

tais que: $[\phi(\vec{r},t), \phi(\vec{r}',t)] = [\pi(\vec{r},t), \pi(\vec{r}',t)] = 0$

$$(323.3)$$

e $[\phi(\vec{r},t), \pi(\vec{r}',t)] = i\hbar\,\delta(\vec{r}-\vec{r}')$

notar: instante tempo t fixo.

· próxima etapa: determinar a relação entre $\phi(x)$ e ops. de criação / destruição de partículas;

inicial: soluções Eq. (323.1), partícula livre:

$$\varphi^{+}_{p}(x) = e^{-ipx/\hbar} = e^{-ip_\mu x^\mu/\hbar} \quad : \text{ sol. energia positiva}$$

(324.1)

$$\varphi^{-}_{p}(x) = e^{+ipx/\hbar} = e^{+ip_\mu x^\mu/\hbar} \quad : \quad \text{"} \quad \text{"} \quad \text{negativa}$$

de fato,

$$\partial^\mu \varphi^{+}_{p}(x) = \frac{\partial}{\partial x_\mu}\left(e^{-ip^\nu x_\nu/\hbar}\right) = -i\frac{p^\mu}{\hbar}\, e^{-ip^\nu x_\nu/\hbar}$$

$$\partial_\mu \partial^\mu \varphi^{+}_{p}(x) = \frac{\partial}{\partial x^\mu}\left(-i\frac{p^\mu}{\hbar}\, e^{-ip_\nu x^\nu/\hbar}\right) = (-i)^2 p_\mu p^\mu\, e^{-ip_\nu x^\nu/\hbar}\,\frac{1}{\hbar^2}$$

↳ Eqs. (323.1) e (324.1) : $\underbrace{\left(-p_\mu p^\mu \cdot \frac{1}{\hbar^2} + \left(\frac{mc}{\hbar}\right)^2\right)}_{=0,\ \text{pois } p_\mu p^\mu = (mc)^2 \ :\ \text{Eq. } (245.1)} \varphi^{+}_{p}(x) = 0$

notar : Eq. (245.2) :

$$E = cp^0 = c\hbar k^0 = \hbar\omega_{\vec{k}} = \left(m^2c^4 + |\vec{p}|^2c^2\right)^{1/2} =$$

$$= \left(m^2c^4 + \hbar^2c^2|\vec{k}|^2\right)^{1/2} = \hbar c\left(\underbrace{\left(\frac{mc}{\hbar}\right)^2}_{\bar{m}^2} + |\vec{k}|^2\right)^{1/2} > 0$$

ou : $k^0 = \frac{1}{c}\,\omega_{\vec{k}} = +\left(\bar{m}^2 + |\vec{k}|^2\right)^{1/2}$

$E = \hbar\omega_k$: energia partícula momento $\vec{p} = \hbar\vec{k}$.

· similar Eq. (294.1), vamos considerar solução geral (323.1), i.e. combinação linear (324.1) : pacote ondas planas :

$$\phi(x) = \phi^{+}(x) + \phi^{-}(x) \tag{324.2}$$

onde

$$\phi^{+}(x) = \sum_{\vec{k}} \left(\frac{\hbar c^2}{2V\omega_{\vec{k}}}\right)^{1/2} a(\vec{k})\, e^{-ikx} = \sum_{\vec{k}} \left(\frac{\hbar c}{2V\omega_{\vec{k}}}\right)^{1/2} a(\vec{k})\, e^{i(\vec{k}\cdot\vec{r} - \omega_{\vec{k}} t)}$$

e

$$\phi^{-}(x) = \sum_{\vec{k}} \left(\frac{\hbar c^2}{2V\omega_{\vec{k}}}\right) a^{+}(\vec{k})\, e^{+ikx} = \sum_{\vec{k}} \left(\frac{\hbar c}{2V\omega_{\vec{k}}}\right)^{1/2} a^{+}(\vec{k})\, e^{-i(\vec{k}\cdot\vec{r} - \omega_{\vec{k}} t)}$$

(325.1)

onde coeficientes $a(\vec{k}) \in \mathbb{C}$.

· quantização canônica:
coef. $a(\vec{k})$ e $a^{+}(\vec{k})$ $\longrightarrow$ operadores $a(\vec{k})$ e $a^{+}(\vec{k})$;

Eqs. (323.3) e (324.2) $\rightarrow$ álgebra ops. $a(\vec{k})$ e $a^{+}(\vec{k})$:

$$[a(\vec{k}), a(\vec{k}')] = [a^{+}(\vec{k}), a^{+}(\vec{k}')] = 0$$

(325.2)

e $[a(\vec{k}), a^{+}(\vec{k}')] = \delta_{\vec{k},\vec{k}'}$

Exercício: verificar Eq. (325.2).

comparar Eqs. (190.1) e (325.2): Eq. (325.2): álgebra ops. criação/destruição bósons;

↳ $a^{+}(k)$: op. criação partícula (bóson) c/ momento $\vec{p} = \hbar\vec{k}$ e energia $E = \hbar\omega_{\vec{k}}$

$\hat{n}(\vec{k}) = a^{+}(\vec{k})\, a(\vec{k})$: op. número de partícula cujos autovalores $n(\vec{k}) = 0, 1, 2, \ldots$

(325.3)

· hamiltoniano correspondente (323.2),

Eq. (317.1): $cP^0 = H = \int d^3r\, \mathcal{H}$,

onde

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L}$$

$$= c^2\pi^2 - \frac{1}{2}\Big(c^2\pi^2 + \underbrace{\partial_\kappa\phi\,\partial^\kappa\phi}_{-(\vec{\nabla}\phi)^2} - \bar{m}^2\phi^2\Big)$$

$$= \frac{1}{2}\left(c^2\pi^2(x) + (\vec{\nabla}\phi(x))^2 + \left(\frac{mc}{\hbar}\right)^2\phi^2\right) \qquad (326.1)$$

Eqs. (324.2) e (326.1), veja pg. 326.1:

$$H = \frac{1}{2}\sum_{\vec{k}} \hbar\omega_{\vec{k}}\left(a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k})\right)$$

$$= \sum_{\vec{k}} \hbar\omega_{\vec{k}}\left(a^\dagger(\vec{k})a(\vec{k}) + 1/2\right) \qquad (326.2)$$

notar: Eq. (326.2) ok c/ interpretação (325.3), i.e,
$\hat{n}(\vec{k})$: op. número partículas c/ momento $\vec{p} = \hbar\vec{k}$.

· similar: momento linear campo, Eq. (317.1):

$$\vec{P} = -\int d^3r\, \pi(x)\vec{\nabla}\phi(x) = -\frac{1}{c^2}\int d^3r\, \dot{\phi}(x)\vec{\nabla}\phi(x)$$

Eqs. (317.1) e (324.2) (Exercício):

$$\vec{P} = \frac{1}{2}\sum_{\vec{k}} \hbar\vec{k}\left(a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k})\right)$$

$$= \sum_{\vec{k}} \hbar\vec{k}\left(a^\dagger(\vec{k})a(\vec{k}) + 1/2\right). \qquad (326.3)$$

· estado fundamental hamiltoniano (326.2):
vácuo $|0\rangle$ p/ partículas $a(\vec{k})$, i.e,

$$a(\vec{k})|0\rangle = 0,\ \text{p/}\ \forall\ \vec{k} \quad \text{ou} \quad \phi^+(x)|0\rangle = 0,\ \text{p/}\ \forall\ x \qquad (326.4)$$

· sobre Eq. (326.2),

Eq. (324.2) : $\phi(x) = \sum_{\vec{k}} \left(\frac{\hbar c^2}{2V\omega_{\vec{k}}}\right)^{1/2} \left(a(\vec{k})\, e^{i(\vec{k}\cdot\vec{r} - \omega_{\vec{k}} t)} + a^{+}(\vec{k})\, e^{-i(\vec{k}\cdot\vec{r} - \omega_{\vec{k}} t)} \right)$

$\hookrightarrow \quad \dot{\phi}(x) = \sum_{\vec{k}} \left(\frac{\hbar c^2}{2V\omega_{\vec{k}}}\right)^{1/2} \left(-i\omega_{\vec{k}}\, a(\vec{k})\, e^{i(\vec{k}\cdot\vec{r} - \omega_{\vec{k}} t)} + i\omega_{\vec{k}}\, a^{+}(\vec{k})\, e^{-i(\vec{k}\cdot\vec{r} - \omega_{\vec{k}} t)} \right)$

$\vec{\nabla}\phi(x) = \sum_{\vec{k}} \left(\frac{\hbar c^2}{2V\omega_{\vec{k}}}\right)^{1/2} \left(i\vec{k}\, a(\vec{k})\, e^{i(\vec{k}\cdot\vec{r} - \omega_{\vec{k}} t)} - i\vec{k}\, a^{+}(\vec{k})\, e^{-i(\vec{k}\cdot\vec{r} - \omega_{\vec{k}} t)} \right)$

$\hookrightarrow \quad H = \frac{1}{2} \sum_{\vec{k}\vec{q}} \left(\frac{\hbar c^2}{2V\omega_{\vec{k}}} \frac{\hbar c^2}{2V\omega_{\vec{q}}} \right)^{1/2} * \Bigg($

$$\Bigg(\underbrace{\frac{(-i\omega_{\vec{k}})(-i\omega_{\vec{q}})}{c^2} + (i\vec{k})(-i\vec{q}) + \bar{m}^2}_{-\frac{\omega_{\vec{k}}^2}{c^2} + \vec{k}^2 + \bar{m}^2 = 0} \Bigg) a(\vec{k})\, a(\vec{q})\, e^{-i(\omega_{\vec{k}} + \omega_{\vec{q}})t} \underbrace{\int d^3r\, e^{i(\vec{k}+\vec{q})\cdot\vec{r}}}_{V\delta_{\vec{k},-\vec{q}}}$$

$$+ \Bigg(\underbrace{\frac{(i\omega_{\vec{k}})(i\omega_{\vec{q}})}{c^2} + (-i\vec{k})(-i\vec{q}) + \bar{m}^2}_{=0} \Bigg) a^{+}(\vec{k})\, a^{+}(\vec{q})\, e^{+i(\omega_{\vec{k}} + \omega_{\vec{q}})t} \underbrace{\int d^3r\, e^{-i(\vec{k}+\vec{q})\cdot\vec{r}}}_{V\delta_{\vec{k},-\vec{q}}}$$

$$+ \Bigg(\underbrace{\frac{(-i\omega_{\vec{k}})(+i\omega_{\vec{q}})}{c^2} + (+i\vec{k})(-i\vec{q}) + \bar{m}^2}_{\frac{\omega_{\vec{k}}^2}{c^2} + \vec{k}^2 + \bar{m}^2 = 2\omega_{\vec{k}}^2/c^2} \Bigg) a(\vec{k})\, a^{+}(\vec{q})\, e^{-i(\omega_{\vec{k}} - \omega_{\vec{q}})t} \underbrace{\int d^3r\, e^{i(\vec{k}-\vec{q})\cdot\vec{r}}}_{V\delta_{\vec{k},\vec{q}}}$$

$$+ \Bigg(\underbrace{\frac{(+i\omega_{\vec{k}})(-i\omega_{\vec{q}})}{c^2} + (-i\vec{k})(+i\vec{q}) + \bar{m}^2}_{= 2\omega_{\vec{k}}^2/c^2} \Bigg) a^{+}(\vec{k})\, a(\vec{q})\, e^{+i(\omega_{\vec{k}} - \omega_{\vec{q}})t} \underbrace{\int d^3r\, e^{-i(\vec{k}-\vec{q})\cdot\vec{r}}}_{V\delta_{\vec{k},\vec{q}}} \Bigg)$$

↳ energia estado fundamental $E_0 = \frac{1}{2}\sum_{\vec{k}} \omega_{\vec{k}} \to +\infty$;

cte E_0 pode ser eliminada se considerarmos energias w.r.t. vácuo!

- alternativa: definir operadores via ordem normal;

Definição: ordem normal ops. criação e destruição: (327.1)

(ops. criação) * (ops. destruição);

notação: $:\text{op.}:$

Exemplo: $:a(\vec{k}_1)\,a^\dagger(\vec{k}_2)\,a(\vec{k}_3)\,a^\dagger(\vec{k}_4): =$

$$= a^\dagger(\vec{k}_2)\,a^\dagger(\vec{k}_4)\,a(\vec{k}_1)\,a(\vec{k}_2)$$

notar: posição ops. modificada considerando que $[a(\vec{k}), a^\dagger(\vec{k}')] = 0$!

Dessa forma, temos que:

· Eq. (326.1): $$H = \int d^3r \, :\frac{1}{2}\left(c^2\pi^2 + (\vec{\nabla}\phi)^2 + \bar{m}^2\phi^2\right):$$

$$= \sum_{k} \hbar\omega_{\vec{k}} \, :\frac{1}{2}\left(a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k})\right):$$

$$= \sum_{k} \hbar\omega_{\vec{k}} \, a^\dagger(\vec{k})a(\vec{k}) \qquad (327.1)$$

· Eq. (326.3): $$\vec{P} = \int d^3r \, :\frac{(-1)}{c}\pi\vec{\nabla}\phi:$$

$$= \sum_{\vec{k}} \hbar\vec{k} \, :\frac{1}{2}\left(a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k})\right):$$

$$= \sum_{k} \hbar\vec{k} \, a^\dagger(\vec{k})a(\vec{k}) \qquad (327.2)$$

Obs.: p/ descrição partículas carregadas, é necessário considerar campo escalar $\phi(x) \in \mathbb{C}$: veja Sec. 3.2. Mandl and Shaw.

· Campo de Dirac.

considerar: campo spinorial (4-componentes) $\psi(x) \in \mathbb{C}$ que satisfaz a eq. de movimento (262.2) : Eq. de Dirac :

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi(x) = 0 \tag{328.1}$$

↳ p/ o campo adjunto $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$, temos que

$$(i\hbar\gamma^\mu\partial_\mu + mc)\bar{\psi}(x) = 0 ; \tag{328.2}$$

considerando as 8-componentes $\psi_\alpha(x)$ e $\bar{\psi}_\alpha(x)$, $\alpha = 1,2,3,4$, como variáveis independentes, verifica-se que as eqs. de movimento (328.1) e (328.2) podem ser derivadas da densidade de lagrangiana (exercício):

$$\mathcal{L} = c\,\bar{\psi}(x)\left(i\hbar\gamma^\mu\partial_\mu - mc\right)\psi(x) \tag{328.3}$$

↳ $\pi_\alpha(x) = \dfrac{\partial\mathcal{L}}{\partial\dot{\psi}_\alpha} = i\hbar\psi^\dagger_\alpha(x)$

: momentos canonicamente conjugados

e $\bar{\pi}_\alpha(x) = \dfrac{\partial\mathcal{L}}{\partial\dot{\bar{\psi}}} = 0$, $\alpha = 1,2,3,4$

notar:

$$\mathcal{L} = i\hbar c\,\bar{\psi}_\lambda\gamma^\mu_{\lambda\beta}\partial_\mu\psi_\beta - mc^2\bar{\psi}_\beta\psi_\beta \quad ; \quad \beta,\lambda = 1,2,3,4$$

$$= i\hbar\,\bar{\psi}_\lambda\gamma^0_{\lambda\beta}\partial_t\psi_\beta + i\hbar c\,\bar{\psi}_\lambda\gamma^k_{\lambda\beta}\partial_k\psi_\beta - mc^2\bar{\psi}_\beta\psi_\beta$$

$\hookrightarrow \pi_\alpha = \dfrac{\partial \mathcal{L}}{\partial \dot{\psi}_\alpha} = i\hbar \bar{\psi}_\lambda \gamma^0_{\lambda\alpha} = i\hbar \psi^+_\alpha$ e $\bar{\pi}_\alpha = \dfrac{\partial \mathcal{L}}{\partial \dot{\bar{\psi}}_\alpha} = 0$

· hamiltoniano correspondente (328.3)

Eq. (317.1) : $cP^0 = H = \int d^3r \, \mathcal{H}$

onde

$$\mathcal{H} = \pi_\alpha \dot{\psi}_\alpha + \bar{\pi}_\alpha \dot{\bar{\psi}}_\alpha - \mathcal{L}$$

$$= i\hbar \psi^+_\alpha \dot{\psi}_\alpha - i\hbar c \bar{\psi} \gamma^\mu \partial_\mu \psi + mc^2 \bar{\psi}\psi$$

$$= i\hbar \psi^+ \dot{\psi} \underbrace{- i\hbar c \bar{\psi} \gamma^0 \partial_0 \psi}_{-i\hbar \psi^+ \dot{\psi}} - i\hbar c \bar{\psi} \gamma^\kappa \partial_\kappa \psi + mc^2 \bar{\psi}\psi$$

$$\hookrightarrow \mathcal{H} = \bar{\psi}(x) \left(-i\hbar c \gamma^\kappa \partial_\kappa + mc^2 \right) \psi(x) \qquad (329.1)$$

ou $H = \int d^3r \, \bar{\psi}(x) \left(-i\hbar c \gamma^\kappa \partial_\kappa + mc^2 \right) \psi(x)$

· momento linear campo, Eq. (317.1) :

$$P^\kappa = \int d^3r \, \pi_\alpha \frac{\partial \psi_\alpha}{\partial x_\kappa} + \bar{\pi}_\alpha \frac{\partial \bar{\psi}_\alpha}{\partial x_\kappa} = \int d^3r \, \pi_\alpha \frac{\partial \psi_\alpha}{\partial x_\kappa}$$

$$\hookrightarrow \vec{P} = \int d^3r \, i\hbar \psi^+_\alpha(x) (-\vec{\nabla} \psi_\alpha(x)) = -i\hbar \int d^3r \, \psi^+(x) \vec{\nabla} \psi(x) \qquad (329.2)$$

· momento angular campo,

Lembrar Eqs. (271.3) e (274.2) :

$$\psi'_\alpha(x') = \psi_\alpha(x) - \frac{i}{4} \Delta\omega_{\mu\nu} \sigma^{\mu\nu}_{\alpha\beta} \psi_\beta(x) \; ; \quad \alpha, \beta = 1,2,3,4 \qquad (329.3)$$

comparando Eqs. (320.1) e (329.3), temos que

$$S^{\mu\nu}_{\alpha\beta} = -\frac{i}{2}\sigma^{\mu\nu}_{\alpha\beta} \quad ; \qquad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \quad : \text{Eq. } (275.3)$$

Eq. (321.2) :

$$cM^{ij} = \int d^3r \; i\hbar c\,\psi^+_\alpha \frac{(-i)}{2}\sigma^{ij}_{\alpha\beta}\psi_\beta + \left(i\hbar\,\psi^+_\alpha x^i \frac{\partial}{\partial x_j}\psi_\alpha - i\hbar\psi^+_\alpha x^j\frac{\partial}{\partial x_i}\psi_\alpha\right)c$$

$$\hookrightarrow \vec{M} = \int d^3r \; \underbrace{\psi^+(x)\frac{\hbar}{2}\hat{\sigma}\psi(x)}_{\text{spin}} + \underbrace{\psi^+(x)\left(\vec{r}\times(-i\hbar\vec{\nabla})\right)\psi(x)}_{\text{momento angular orbital}} \qquad (330.1)$$

onde $\hat{\sigma} = (\sigma^{23}, \sigma^{31}, \sigma^{12})$: veja Eq. (321.2)

· transformação global de gauge (321.3).
como a densidade de lagrangiana (328.3) é invariante sob transf. (321.3), temos que carga Q é conservada:

$$\text{Eq. } (322.2): \quad Q = -\frac{i}{\hbar} q \int d^3r \; \underbrace{\pi_\alpha(x)\psi_\alpha(x)}_{i\hbar\psi^+_\alpha\psi_\alpha} - \underbrace{\pi^+_\alpha(x)\psi^+_\alpha(x)}_{0}$$

$$\hookrightarrow Q = q\int d^3r \; \psi^+(x)\psi(x) \qquad (330.2)$$

· quantização campo de Dirac.
Lembrar: soluções (286.1), Eq. de Dirac, partícula livre:

$$\psi^{(+)}_r(x) = u_r(\vec{p})\,e^{-ip^\mu x_\mu/\hbar} \quad : \text{energia positiva}$$

$r = 1, 2$

$$\psi^{(-)}_r(x) = v_r(\vec{p})\,e^{+ip^\mu x_\mu/\hbar} \quad : \text{energia negativa}$$

onde spinores $u_n(p)$ e $v_n(p)$ satisfazem as relações de ortonormalidade (288.1):

$$u^{\dagger}_n(p)\, u_s(p) = v^{\dagger}_n(p)\, v_s(p) = \frac{E_{\vec{p}}}{mc^2}\, \delta_{n,s}$$

(331.1)

e $u^{\dagger}_n(p_0, \vec{p})\, v_s(p_0, -\vec{p}) = 0$; $\quad E_{\vec{p}} = \left(|\vec{p}|^2 c^2 + m^2 c^4\right)^{1/2} > 0$

e são autoestados do operador helicidade

$$\sigma_{\vec{p}} = \frac{\hat{\sigma} \cdot \vec{p}}{|\vec{p}|} \quad , \quad \hat{\sigma} = (\sigma^{23}, \sigma^{31}, \sigma^{12}) \ , \ i.e.,$$

(331.2)

$$\sigma_{\vec{p}}\, u_n(p) = (-1)^{n+1} u_n(p) \quad e \quad \sigma_{\vec{p}}\, v_n(p) = (-1)^n v_n(p), \ n = 1, 2$$

similar quantização campo Klein-Gordon, vamos considerar solução geral (328.1), combinação linear soluções (286.1) : pacote de ondas planas (294.1):

$$\psi(x) = \psi^{+}(x) + \psi^{-}(x)$$

$$= \sum_{n=1,2} \sum_{\vec{p}} \left(\frac{mc^2}{VE_{\vec{p}}}\right)^{1/2} \left(c_n(p)\, u_n(p)\, e^{-ipx/\hbar} + d^{\dagger}_n(p)\, v_n(p)\, e^{+ipx/\hbar} \right)$$

e (331.3)

$$\bar{\psi}(x) = \bar{\psi}^{+}(x) + \bar{\psi}^{-}(x)$$

$$= \sum_{n=1,2} \sum_{\vec{p}} \left(\frac{mc^2}{VE_{\vec{p}}}\right)^{1/2} \left(d_n(p)\, \bar{v}_n(p)\, e^{-ipx/\hbar} + c^{\dagger}_n(p)\, \bar{u}_n(p)\, e^{+ipx/\hbar} \right)$$

onde coeficientes $c_n(p)$ e $d_n(p) \in \mathbb{C}$;
similar procedimento considerado p/ campo de Klein-Gordon, porém como férmions ~ estatística de Fermi-Dirac, temos que:

coefs. $C_n(p)$ e $d_n(p)$ $\longrightarrow$ operadores $C_n(\vec{p})$, $C_n^+(\vec{p})$

$d_n(\vec{p})$, $d_n^+(\vec{p})$, tais que

$$\{C_n(\vec{p}), C_s^+(\vec{p}')\} = \{d_n(\vec{p}), d_s^+(\vec{p}')\} = \delta_{n,s}\,\delta_{\vec{p},\vec{p}'}$$

$$\{C_n(\vec{p}), C_s(\vec{p}')\} = \{C_n^+(\vec{p}), C_s^+(\vec{p}')\} = 0$$

(332.1)

$$\{d_n(\vec{p}), d_s(\vec{p}')\} = \{d_n^+(\vec{p}), d_s^+(\vec{p}')\} = 0$$

$$\{C_n(\vec{p}), d_s(\vec{p}')\} = \{C_n(\vec{p}), d_s^+(\vec{p}')\} = 0 \quad :$$

: dois conjuntos ops. fermiônicos independentes =

= dois tipos de partículas!

↳ $\hat{n}_n^+(\vec{p}) = C_n^+(\vec{p})\,C_n(\vec{p})$

: ops. número de partículas

$\hat{n}_n^-(\vec{p}) = d_n^+(\vec{p})\,d_n(\vec{p})$ (332.2)

· o estado de vácuo $|0\rangle$ é definido tal que:

$$C_n(\vec{p})|0\rangle = d_n(\vec{p})|0\rangle = 0 \quad , \ p/ \ \forall \ \vec{p}$$

ou (332.3)

$$\psi^+(x)|0\rangle = \bar{\psi}^+(x)|0\rangle = 0 \quad , \ p/ \ \forall \ x$$

· Eqs. (331.3) e (332.1): verifica-se que (exercício):

$$\{\psi_\alpha(\vec{r},t), \psi_\beta^+(\vec{r}',t)\} = \delta_{\alpha,\beta}\,\delta(\vec{r}-\vec{r}')$$

(332.4)

$$\{\psi_\alpha(\vec{r},t), \psi_\beta(\vec{r}',t)\} = \{\psi_\alpha^+(\vec{r},t), \psi_\beta^+(\vec{r}',t)\} = 0 \quad :$$

: comparar c/ Eq. (323.3).

· sobre as cte.s de movimento.

similar campo de Klein-Gordon. quantidades definidas w.r.t. vácuo

↳ necessário definir operadores via ordem normal ops. criação e destruição;

Definição ordem normal p/ ops. fermiônicos : similar Eq. (327.1) porém, nesse caso, posição ops. modificada considerando que todos anticomutadores são nulos!

Exemplo: $:c_r(\vec{p}_1)\, c_r^{\dagger}(\vec{p}_2)\, c_r(\vec{p}_3)\, c_r(\vec{p}_4): =$

$$= -\, c_r^{\dagger}(\vec{p}_2)\, c_r(\vec{p}_1)\, c_r(\vec{p}_3)\, c_r(\vec{p}_4)$$

Dessa forma, temos que

Eq. (329.1): $H = \int d^3r \; :\bar{\psi}(x)\left(-i\hbar c\, \gamma^k \partial_k + mc^2\right)\psi(x):$

$$= \sum_{r=1,2} \sum_{\vec{p}} E_{\vec{p}} \left(c_r^{\dagger}(\vec{p})\, c_r(\vec{p}) + d_r^{\dagger}(\vec{p})\, d_r(\vec{p}) \right) \qquad (333.1)$$

Eq. (329.2): $\vec{P} = -i\hbar \int d^3r \; :\psi^{\dagger}(x)\, \vec{\nabla}\psi(x):$

$$= \sum_{r=1,2} \sum_{\vec{p}} \vec{p} \left(c_r^{\dagger}(\vec{p})\, c_r(\vec{p}) + d_r^{\dagger}(\vec{p})\, d_r(\vec{p}) \right)$$

(333.2)

Eq. (330.2):

$$Q = q \int d^3r \; :\psi^{\dagger}(x)\, \psi(x):$$

$$= -e \sum_{r=1,2} \sum_{\vec{p}} \left(c_r^{\dagger}(\vec{p})\, c_r(\vec{p}) - d_r^{\dagger}(\vec{p})\, d_r(\vec{p}) \right)$$

(333.3)

onde $q = -e < 0$

notar Eqs. (333.1) - (333.3) : se m = massa do elétron p/ Eq. (328.3)
↳ interpretação:

$c_r^+(\vec{p})$: op. criação elétron, momento $\vec{p}$, energia $E_{\vec{p}}$

$d_r^+(\vec{p})$: " " pósitron, " " , " "

· sobre o spin das partículas,
considerar o primeiro termo Eq. (330.1):

$$\vec{S} = \int d^3r\, \psi^+(x) \frac{\hbar}{2} \hat{\sigma}\, \psi(x)$$

$$\hookrightarrow S_{\vec{p}} = \frac{\vec{S}\cdot\vec{p}}{|\vec{p}|} = \frac{\hbar}{2}\int d^3r\, \psi^+(x)\, \frac{\hat{\sigma}\cdot\vec{p}}{|\vec{p}|}\, \psi(x) = \frac{\hbar}{2}\int d^3r\, \psi^+(x)\, \sigma_{\vec{p}}\, \psi(x) :$$

: operador helicidade (334.1)

notar : $S_{\vec{p}}$ ~ spin partícula, direção longitudinal do movimento, i.e. $/\!/ \hat{p}$;

verifica-se que

$$S_{\vec{p}}\, c_r^+(\vec{p})|0\rangle = (-1)^{r+1} \frac{\hbar}{2}\, c_r^+(\vec{p})|0\rangle$$
(334.2)
$$S_{\vec{p}}\, d_r^+(\vec{p})|0\rangle = (-1)^{r+1} \frac{\hbar}{2}\, d_r^+(\vec{p})|0\rangle \quad , r = 1,2,$$

i.e. p/ elétron e pósitron, temos que

projeção spin = $+\hbar/2$, $r = 1$
$-\hbar/2$, $r = 2$: ok c/ Eq. (292.5) !

· notar: estado fundamental hamiltoniano (333.1):
vácuo : $|0\rangle$
e $E_{GS} = 0$

· sobre o teorema spin-estatística,
considerar: procedimento de quantização descrito acima,
porém, álgebra ops. $c_n(\vec{p})$ e $d_n(\vec{p})$:
= " " criação / destruição bósons ;

nesse caso, verifica-se que Eq. (333.1) assume a forma

$$H = \sum_{n=1,2} \sum_{\vec{p}} E_{\vec{p}} \big(\underbrace{c_n^\dagger(\vec{p})\, c_n(\vec{p})}_{\hat{n}_n^+(\vec{p})} - \underbrace{d_n^\dagger(\vec{p})\, d_n(\vec{p})}_{\hat{n}_n^-(\vec{p})} \big) \qquad (335.1)$$

como autovalores $n_n^+(\vec{p}), n_n^-(\vec{p}) = 0, 1, 2, \ldots$
↳ H não apresenta estado de menor energia : NOT OK!

- contradições também são encontradas se a quantização do campo de Klein-Gordon é realizada via ops. criação / destruição férmions, veja Sec. 4.3, Mandl and Shaw;

- resultados acima podem ser generalizados p/ sistemas interagentes e partículas c/ ∀ valor spin
↳ teorema spin-estatística.

· detalhes hamiltoniano (333.1),
notar:
· $(-i\hbar\gamma^\alpha \partial_\alpha + mc)\, u_n(p)\, e^{-ip_\mu x^\mu/\hbar}$

$= e^{-ip_\mu x^\mu/\hbar} (-\gamma^\alpha p_\alpha + mc)\, u_n(p)$

$$= e^{-ip_\mu x^\mu/\hbar}\left(\underbrace{(-\gamma^\mu p_\mu + mc)u_n(p)}_{=0\ :\ \text{Eq. }(286.3)} + \gamma^0 p_0 u_n(p)\right) = e^{-ip_\mu x^\mu/\hbar}\gamma^0 p_0 u_n(p)$$

$$\cdot\ (-i\hbar\gamma^\kappa\partial_\kappa + mc)v_n(p)e^{+ip_\mu x^\mu/\hbar}$$

$$= e^{+ip_\mu x^\mu/\hbar}(+\gamma^\kappa p_\kappa + mc)v_n(p) = e^{ipx/\hbar}\left(\underbrace{(\gamma^\mu p_\mu + mc)v_n(p)}_{=0\ :\ \text{Eq. }(286.4)} - \gamma^0 p_0 v_n(p)\right)$$

Eqs. (329.1) e (333.1) :

$$H = \sum_{n,s}\sum_{\vec{p},\vec{\kappa}}\left(\frac{mc^2}{VE_{\vec{p}}}\cdot\frac{mc^2}{VE_{\vec{\kappa}}}\right)^{1/2} \cdot \int d^3_n\, c$$

$$\cdot\left(d_n(\vec{p})\bar{v}_n(p)e^{-ipx/\hbar} + c^\dagger_n(\vec{p})\bar{u}_n(p)e^{ipx/\hbar}\right)\cdot$$

$$\cdot\left(c_s(\vec{\kappa})\gamma^0\kappa_0 u_s(\kappa)e^{-i\kappa x/\hbar} - d^\dagger_s(\kappa)\gamma^0\kappa_0 v_s(\kappa)e^{i\kappa x/\hbar}\right)$$

$$= \sum_{n,s}\sum_{\vec{p},\vec{\kappa}}\left(\frac{mc^2}{VE_{\vec{p}}}\cdot\frac{mc^2}{VE_{\vec{\kappa}}}\right)^{1/2} \cdot c\cdot\Big($$

$$\kappa_0 d_n(\vec{p})c_s(\vec{\kappa})\underbrace{\bar{v}_n(p)\gamma^0 u_s(\kappa)}_{v^\dagger_n(p_0,\vec{p})u_s(p_0,-\vec{p})=0} e^{-i(E_{\vec{p}}+E_{\vec{\kappa}})t/\hbar}\underbrace{\int d^3_n\, e^{i(\vec{p}+\vec{\kappa})\cdot\vec{n}/\hbar}}_{V\delta_{\vec{p},-\vec{\kappa}}}$$

$$-\kappa_0 c^\dagger_n(\vec{p})d^\dagger_s(\vec{\kappa})\underbrace{\bar{u}_n(p)\gamma^0 v_s(\kappa)}_{u^\dagger_n(p_0,\vec{p})v_s(p_0,-\vec{p})=0} e^{+i(E_{\vec{p}}+E_{\vec{\kappa}})t/\hbar}\underbrace{\int d^3_n\, e^{-i(\vec{p}+\vec{\kappa})\cdot\vec{n}/\hbar}}_{V\delta_{\vec{p},-\vec{\kappa}}}$$

$$+ \kappa_0\, C^\dagger_n(\vec{p})\, C_s(\vec{k})\, \underbrace{\bar{u}_n(p)\, \gamma^0 u_s(k)}_{u^\dagger_n(p)\, u_s(p) = \frac{E_{\vec{p}}}{mc^2}\delta_{n,s}}\, e^{+i(E_{\vec{p}} - E_{\vec{k}})t/\hbar} \underbrace{\int d^3 r\, e^{-i(\vec{p}-\vec{k})\cdot \vec{r}/\hbar}}_{V \delta_{\vec{p},\vec{k}}}$$

$$- \kappa_0\, d_n(\vec{p})\, d^\dagger_s(\vec{k})\, \underbrace{\bar{v}_n(p)\, \gamma^0 v_s(k)}_{v^\dagger_n(p)\, v_s(p) = \frac{E_{\vec{p}}}{mc^2}\delta_{n,s}}\, e^{-i(E_{\vec{p}} - E_{\vec{k}})t/\hbar} \underbrace{\int d^3 r\, e^{+i(\vec{p}-\vec{k})\cdot \vec{r}/\hbar}}_{V \delta_{\vec{p},\vec{k}}}$$

$$H = \sum_n \sum_{\vec{p}} \frac{mc^2}{V E_{\vec{p}}} \cdot \frac{E_{\vec{p}}}{mc^2} V \cdot c\kappa_0\, +$$

$$* \left(C^\dagger_n(\vec{p})\, C_n(\vec{p}) - d_n(\vec{p})\, d^\dagger_n(\vec{p}) \right)$$

· Campo eletromagnético.

inicial: revisão eletrodinâmica;

· Eqs. de Maxwell, vácuo, sistema gaussiano:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\times\vec{B} - \frac{1}{c}\partial_t \vec{E} = \frac{4\pi}{c}\vec{J}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\partial_t \vec{B} = 0 \tag{337.1}$$

vamos considerar, entretanto o sistema de unidades Lorentz-Heaviside; nesse caso, temos que

$$\vec{\nabla}\cdot\vec{E} = \rho \qquad \vec{\nabla}\times\vec{B} - \frac{1}{c}\partial_t \vec{E} = \frac{1}{c}\vec{J}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\partial_t \vec{B} = 0 \tag{337.2}$$

onde $\vec{E}_{L-H} = \frac{1}{\sqrt{4\pi}} \vec{E}_{GAUSS}$ $\vec{B}_{L-H} = \frac{1}{\sqrt{4\pi}} \vec{B}_{GAUSS}$

carga: $e_{L-H} = \sqrt{4\pi}\, e_{GAUSS}$

· as eqs. de Maxwell homogêneas → campos $\vec{E}$ e $\vec{B}$ podem ser escritos em termos potenciais escalar $\Phi = \Phi(\vec{r},t)$ e $\vec{A} = \vec{A}(\vec{r},t)$:

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\partial_t \vec{A} \quad \text{e} \quad \vec{B} = \vec{\nabla}\times\vec{A}\,; \qquad (338.1)$$

notar:

se $\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\Lambda$: transf. de
e $\Phi \to \Phi' = \Phi - \partial_t \Lambda$; $\Lambda = \Lambda(\vec{r},t)$ gauge

(338.2)

↳ $\vec{E}' = -\vec{\nabla}\Phi' - \frac{1}{c}\partial_t \vec{A}' = -\vec{\nabla}\Phi - \frac{1}{c}\partial_t\vec{A} = \vec{E}$

e $\vec{B}' = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} = \vec{B}$, i.e.

conjuntos $(\Phi, \vec{A})$ e $(\Phi', \vec{A}')$ → mesmos $\vec{E}$ e $\vec{B}$: invariancia de gauge

· eqs. de Maxwell não-homogeneas em termos Φ e $\vec{A}$; verifica-se que (exercício):

$$\nabla^2\Phi + \frac{1}{c}\partial_t(\vec{\nabla}\cdot\vec{A}) = -\rho$$

(338.3)

$$\Box\vec{A} + \vec{\nabla}\left(\frac{1}{c}\partial_t\Phi + \vec{\nabla}\cdot\vec{A}\right) = \frac{1}{c}\vec{J}$$: eqs. de movimento p/ Φ e $\vec{A}$

onde $\Box \equiv \frac{1}{c^2}\partial_t^2 - \nabla^2$

notar: 4 eqs. diferenciais (337.2) acopladas 1ª ordem → 2 eqs. diferenciais (338.3) acopladas 2ª ordem

• 2 casos

(i) gauge de Lorentz;
nesse caso, liberdade (338.2) permite escolha:

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\partial_t\Phi = 0$$

↳ Eq. (338.3): $\square\Phi = \frac{1}{c^2}\partial_t^2\Phi - \nabla^2\Phi = \rho$

$$\text{e} \quad \square\vec{A} = \frac{1}{c^2}\partial_t^2\vec{A} - \nabla^2\vec{A} = \frac{1}{c}\vec{J} \qquad (339.1)$$

(ii) gauge de Coulomb (transverso ou radiação);
nesse caso, liberdade (338.2) permite escolha:

$$\vec{\nabla}\cdot\vec{A} = 0$$

↳ Eq. (338.3): $\nabla^2\Phi = -\rho$

$$\square\vec{A} = \frac{1}{c}\vec{J} - \frac{1}{c}\vec{\nabla}(\partial_t\Phi) \qquad (339.2)$$

• considerar: gauge de Coulomb ⊕ ausência fontes: $\rho = \vec{J} = 0$, i.e., campos EM livres;

↳ Eq. (339.2): $\nabla^2\Phi = 0$
como $\lim_{r\to+\infty}\Phi(\vec{r},t) = 0$, podemos considerar $\Phi(\vec{r},t) = 0$;

↳ $\square\vec{A} = \frac{1}{c^2}\partial_t^2\vec{A} - \nabla^2\vec{A} = 0 \rightarrow \vec{A}(\vec{r},t) = A_0 e^{\pm i(\vec{k}\cdot\vec{r}-\omega t)}$

$$\text{e} \quad \omega = c|\vec{k}| \qquad (339.3)$$

como $\vec{\nabla}\cdot\vec{A} = 0 \rightarrow \vec{k}\cdot\vec{A} = 0$: $\vec{A} \perp$ vetor de propagação $\vec{k}$,
i.e., $\vec{A} = \vec{A}(\vec{r},t)$: onda plana transversal

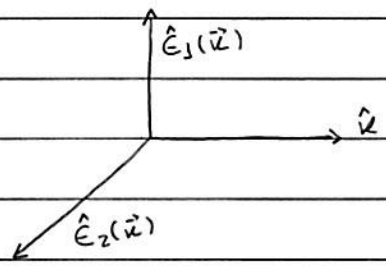
p/ condições de contorno periódicas, i.e.,

$$\vec{A}(x,y,z,t) = \vec{A}(x+L,y,z,t) \; ; \; \dots$$

↳ $\vec{k} = \frac{2\pi}{L}(n_1, n_2, n_3)$; $n_i = 0, \pm 1, \pm 2, \dots$

e versores $\hat{\varepsilon}_1(\vec{k})$ e $\hat{\varepsilon}_2(\vec{k})$ tais que

$$\hat{\varepsilon}_n(\vec{k}) \cdot \hat{\varepsilon}_s(\vec{k}) = \delta_{n,s} \; ; \qquad \hat{\varepsilon}_n(\vec{k}) \cdot \vec{k} = 0 \qquad (340.1)$$



e $\hat{\varepsilon}_n(\vec{k}) \cdot \hat{\varepsilon}_s(-\vec{k}) = (-1)^n \delta_{n,s}$

↳ solução geral Eq. (339.3):

$$\vec{A}(\vec{r},t) = \sum_{n=1,2} \sum_{\vec{k}} \left(\frac{\hbar c^2}{2V\omega_{\vec{k}}} \right)^{1/2} \hat{\varepsilon}_n(\vec{k}) \Big(a_n(\vec{k})\, e^{i(\vec{k}\cdot\vec{r} - \omega_{\vec{k}} t)} + a_n^*(\vec{k})\, e^{-i(\vec{k}\cdot\vec{r} - \omega_{\vec{k}} t)} \Big) \qquad (340.2)$$

onde $\omega_{\vec{k}} = c|\vec{k}|$; coeficientes $a_n(\vec{k}) \in \mathbb{C}$;

$n = 1,2$: estados de polarização e $\vec{A}(\vec{r},t) \in \mathbb{R}$

· formulação covariante EM:

$$F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu = A^{\mu,\nu} - A^{\nu,\mu}$$: tensor de campo (cuidado: definição ≠ Jackson)

onde $A^\mu(x) = (\Phi, \vec{A})$

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \quad (341.1)$$

$$\text{e} \quad F_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$

nesse caso, Eqs. de Maxwell (337.2) assumem a forma:

$$\partial_\nu F^{\mu\nu} = \frac{1}{c} J^\mu$$

$$\text{e} \quad \partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0 \quad (341.2)$$

onde $J^\mu(x) = (c\rho, \vec{J})$ e a eq. de continuidade é dada por:

$$\partial_\mu J^\mu(x) = 0 \quad (341.3)$$

· formalismo lagrangiano.

identificando: $\phi_n(x) \to A^\mu(x)$

$\partial_\nu \phi_n(x) \to \partial_\nu A^\mu(x)$

↳ Eq. (312.2): $\dfrac{\partial \mathcal{L}}{\partial A^\mu} - \partial^\nu \left(\dfrac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} \right) = 0$: Eq. de Lagrange (341.4)

temos que

$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu$: densidade de lagrangiana campo EM (341.5)

Verificar: Eqs. (341.4) e (341.5) → Eq. (341.2), não-homogênea em particular, na ausência de fontes:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}(E^2 - B^2) \qquad (342.1)$$

momento canonicamente conjugado (313.3):

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = \frac{1}{c} \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = -\frac{1}{c} F^{\mu 0},$$

pois $\mathcal{L} = -\frac{1}{4}\left(\partial^\beta A^\alpha - \partial^\alpha A^\beta\right)\left(\partial_\beta A_\alpha - \partial_\alpha A_\beta\right)$

↳ $\pi^0(x) = -\frac{1}{c} F^{00} = 0$

$$\pi^k(x) = -\frac{1}{c} F^{k0} = \frac{1}{c} E^k \overset{\text{Eq. (338.1)}}{=} \frac{1}{c}\partial^k \phi - \frac{1}{c^2}\dot{A}^k \qquad (342.2)$$

↳ densidade de hamiltoniana (313.3):

$$\mathcal{H} = \pi^k \dot{A}_k - \mathcal{L} = -\pi^k \dot{A}^k - \mathcal{L}$$

$$= -\frac{1}{c} E^k \left(c\,\partial^k \phi - c E^k \right) - \frac{1}{2}(E^2 - B^2)$$

$$= \frac{1}{2}(E^2 + B^2) - E^k \partial^k \phi$$

↳ $cP^0 = H = \int d^3 r\, \mathcal{H} = \frac{1}{2} \int d^3 r\, (E^2 + B^2)$: Eq. (317.1) $\qquad (342.3)$

Obs.: $\int d^3 r\, E^k \partial^k \phi = -\int d^3 r\, \phi \underbrace{\partial^k E^k}_{=0} = 0.$

· momento linear campo, Eq. (317.1)

$$P^k = \int d^3 r \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r} \frac{\partial \phi_r}{\partial x_k} = \int d^3 r \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} \frac{\partial A^\mu}{\partial x_k} = \int d^3 r\, \pi_\mu \partial^k A^\mu$$

↳ $P^k = \int d^3r\, \pi_i \partial^k A^i = \int d^3r\, \pi_i (\partial^k A^i - \partial^i A^k + \partial^i A^k)$

$$= \int d^3r\, -\frac{1}{c} F_{i0} F^{ik} + \underbrace{\pi^i \partial^i A^k}_{=0,\ \text{veja Eq. (342.3)}}$$

↳ $P^k = \frac{1}{c} \int d^3r\, F_{0i} F^{ik}$

↳ $\vec{P} = \frac{1}{c} \int d^3r\, \vec{E} \times \vec{B}$ (343.1)

· notar: $P^1 = \frac{1}{c} \int d^3r\, \underbrace{F_{02} F^{21} + F_{03} F^{31}}_{(-E_y)(-B_z) + (-E_z)(B_y) = (E_y B_z - E_z B_y)} = \frac{1}{c} \int d^3r\, (\vec{E} \times \vec{B})_1$

· momento angular campo, Eq. (321.1)

$$c M^{\mu\nu} = \int d^3r\, c\, \pi_\alpha S^{\mu\alpha}_{\alpha\beta} A^\beta + x^\mu T^{0\nu} - x^\nu T^{0\mu} \quad (343.2)$$

em particular,

$$c M^{ij} = \int d^3r\, c\, \pi_\alpha S^{ij}_{\alpha\beta} A^\beta + c(x^i p^j - x^j p^i) \quad (343.3)$$

como, p/ notação componentes 3-vetor Eq. (317.2) assume a forma

$$A'^r(x') = A^r(x) + \frac{1}{2} \Delta\omega_{ij} S^{ij}_{rs} A^s(x)$$

onde $S^{ij}_{rs} = g^{ir} g_s{}^j - g^i{}_s g^{jr} \to S^{ij}_{ij} = -1$ e $S^{ij}_{ji} = +1$

temos que:

$$M^{ij} = \int d^3x \, \underbrace{(-1)(\pi_i A^j - \pi_j A^i)}_{\text{spin}} + \underbrace{(x^i p^j - x^j p^i)}_{\text{momento ang. orbital}} \qquad (343.4)$$

· quantização campo EM.

em princípio, similar procedimento de quantização canônica campo de Klein-Gordon, Eq. (323.3), i.e.

4-variáveis dinâmicas $A^\mu(x)$ (campos clássicos) e $\pi^\kappa(x)$ $\longrightarrow$ operadores $A^\mu(x)$ e $\pi^\kappa(x)$, tais que:

$$[A^\mu(\vec{r},t), A^\nu(\vec{r}',t)] = 0 \qquad [\pi^i(\vec{r},t), \pi^j(\vec{r}',t)] = 0$$

$$[A_0(\vec{r},t), \pi^\kappa(\vec{r}',t)] = 0 \qquad : \text{instante } t\text{-fixo!}$$

$$\text{e} \quad [A_i(\vec{r},t), \pi^j(\vec{r}',t)] = -[A^i(\vec{r},t), \pi^j(\vec{r}',t)] = i\hbar\,\delta_{ij}\,\delta(\vec{r}-\vec{r}') \qquad (344.1)$$

Obs.: lembrar que π^κ é canon. conjugado a A_κ : Eq. (342.2)

· como $\pi^0(x) = 0$ e $A^0(x)$ comuta com todos ops.
↳ podemos considerar $A^0(x) = \text{cte} \in \mathbb{C}$!

· notar:

$$[\pi^i(\vec{r},t), A^j(\vec{r}',t)] = \frac{1}{c}[E^i(\vec{r},t), A^j(\vec{r}',t)] = i\hbar\,\delta_{ij}\,\delta(\vec{r}-\vec{r}') \qquad (344.2)$$

$\sum_{i=1}^{3} \partial_i$:

$$\underbrace{\partial_i E^i(\vec{r},t)}_{=0} A^j(\vec{r}',t) - A^j(\vec{r}',t)\underbrace{\partial_i E^i(\vec{r},t)}_{=0} = \underbrace{i\hbar \sum_{i=1}^{3} \delta_{ij}\,\partial_i\,\delta(\vec{r}-\vec{r}')}_{(I)\,\neq\,0},$$

i.e. relação de comutação (344.2) não é consistente c/ lei de Gauss!

notar: $(I) = \partial_j \delta(\vec{r}-\vec{r}') = \dfrac{\partial}{\partial x^j}\displaystyle\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$

$$= \int \frac{d^3k}{(2\pi)^3}\, i k_j\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

↳ p/ $(I)=0$ devemos, de fato, considerar:

$$\delta_{ij}\delta(\vec{r}-\vec{r}') \longrightarrow \delta_{ij}^{Tr}(\vec{r}-\vec{r}') \equiv \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) :$$

: "função δ transversal" (345.1)

de fato:

$$\sum_i \partial_i \delta_{ij}^{Tr}(\vec{r}-\vec{r}') = \sum_i \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\, i\left(\delta_{ij}k_i - \frac{k_i k_i k_j}{k^2}\right) = 0$$

dessa forma, Eq. (344.2) deve ser reescrito como:

$$[\pi^i(\vec{r},t), A^j(\vec{r}',t)] = +i\hbar\, \delta_{ij}^{Tr}(\vec{r}-\vec{r}') \qquad (345.2)$$

· procedimento similar ao discutido acima:
Eq. (345.2) $\rightarrow \vec{\nabla}\cdot\vec{A} = $ cte $\in \mathbb{C}$!

· como $A^0 = \phi = $ cte e $\vec{\nabla}\cdot\vec{A} = $ cte
↳ ϕ e $\vec{k}\cdot\vec{A}$ não são variáveis dinâmicas;

em particular, podemos escolher gauge tal que
$\phi = \vec{\nabla}\cdot\vec{A} = 0$: ok c/ gauge de Coulomb (339.2);

nesse caso:
(i) invariância teoria por transfs. Lorentz e gauge não é explícita;
(ii) somente 2 modos são variáveis dinâmicas: componentes transversais w.r.t. $\hat{k}$.

próxima etapa: quantização campo EM no gauge de Coulomb;

Obs.: veja, e.g., Cap. 5, Mandl and Shaw p/ quantização covariante campo EM.

· similar campo de Klein-Gordon, vamos considerar expansão (340.2) do potencial vetor $\vec{A}(\vec{r},t)$ em termos ondas planas = solução campo EM livre;

⊕ coef. $a_n(\vec{k})$ e $a_n^*(\vec{k})$ → operadores $a(\vec{k})$ e $a^\dagger(\vec{k})$;

Eqs. (340.2), (344.1) e (345.2) → operadores $a(\vec{k})$ e $a^\dagger(\vec{k})$ satisfazem álgebra ops. criação/destruição bosons, i.e., (veja pg.):

$$[a_n(\vec{k}), a_s(\vec{k}')] = [a_n^\dagger(\vec{k}), a_s^\dagger(\vec{k}')] = 0$$

$$\text{e} \quad [a_n(\vec{k}), a_s^\dagger(\vec{k}')] = \delta_{n,s}\,\delta_{\vec{k},\vec{k}'} \tag{346.1}$$

Dessa forma:

$$\vec{A}(\vec{r},t) = \sum_{n=1,2}\sum_{\vec{k}} \left(\frac{\hbar c^2}{2V\omega_{\vec{k}}}\right)^{1/2} \hat{\varepsilon}_n(\vec{k}) \left(a_n(\vec{k})\, e^{-ik\cdot x} + a_n^\dagger(\vec{k})\, e^{+ik\cdot x} \right) \tag{346.2}$$

onde $\omega_{\vec{k}} = c|\vec{k}|$ e $k\cdot x = k^\mu x_\mu = \omega_{\vec{k}} t - \vec{k}\cdot\vec{r}$

interpretação:

$a_n^\dagger(\vec{k})$: operador criação partícula (fóton), momento $\hbar\vec{k}$, energia $\hbar\omega_{\vec{k}}$ e polarização $n = 1,2$;

de fato, verifica-se que:

Eq. (342.3) : $H = \frac{1}{2}\int d^3r : E^2 + B^2 :$

$$= \frac{1}{2}\int d^3r : \frac{1}{c^2}\dot{A}^2 + (\vec{\nabla}\times\vec{A})^2 : \quad , \text{ pois } \vec{E} = -\frac{1}{c}\dot{\vec{A}} \text{ no gauge de Coulomb}$$

$$= \sum_{n=1,2}\sum_{\vec{k}} \hbar\omega_{\vec{k}}\, a^{\dagger}_n(\vec{k})\, a_n(\vec{k}) \qquad (347.1)$$

e

Eq. (343.1) : $\vec{P} = \frac{1}{c}\int d^3r : \vec{E}\times\vec{B} :$

$$= -\frac{1}{c}\int d^3r : \dot{A}^i \vec{\nabla} A^i : \quad , \text{ gauge de Coulomb}$$

$$= \sum_{n=1,2}\sum_{\vec{k}} \hbar\vec{k}\, a^{\dagger}_n(\vec{k})\, a_n(\vec{k}) \qquad (347.2)$$

· notar:

(i) estado fundamental H : vácuo $|0\rangle$:

$: a_n(\vec{k})|0\rangle = 0$, p/ $\forall\, \vec{k}$ e $n = 1,2$;

(ii) como $\vec{A}(\vec{r},t) \in \mathbb{R}$ -> carga elétrica fóton = 0;

(iii) similaridade entre partículas ~ campos Klein-Gordon e E.M., entretanto:

Klein-Gordon: massa partícula $m \neq 0$: $p_\mu p^\mu = (mc)^2$

E.M. : " " $m = 0$: $p_\mu p^\mu = 0$

(iv) como $\vec{A}(\vec{r},t)$: campo vetorial -> spin fóton = 1;

como $\vec{k}\cdot\vec{A} = 0$ -> apenas projeções ± 1 // $\hat{k}$ são possíveis!

· sobre a álgebra (346.1).

Eq. (346.2):

$$\vec{A}(\vec{r},t) = -i \sum_{n=1,2} \sum_{\vec{k}} \left(\frac{\hbar c^2 \omega_{\vec{k}}}{2V}\right)^{1/2} \hat{\epsilon}_n(\vec{k}) \left(a_n(\vec{k}) e^{-ik\cdot x} - a^\dagger_n(\vec{k}) e^{+ik\cdot x}\right)$$

(348.1)

· $\int d^3r\, e^{ip\cdot x}\, \hat{\epsilon}_s(\vec{p})\,\cdot$ Eqs. (346.2):

$$\int d^3r\, e^{ip\cdot x}\, \hat{\epsilon}_s(\vec{p})\cdot\vec{A}(\vec{r},t) = \sum_{n=1,2}\sum_{\vec{k}} \left(\frac{\hbar c^2}{2V\omega_{\vec{k}}}\right)^{1/2} \hat{\epsilon}_s(\vec{p})\cdot\hat{\epsilon}_n(\vec{k}) \,*$$

$$* \Big(a_n(\vec{k})\, e^{-i(\omega_{\vec{k}} - \omega_{\vec{p}})t} \underbrace{\int d^3r\, e^{i(\vec{k}-\vec{p})\cdot\vec{r}}}_{V\delta_{\vec{k},\vec{p}}} +$$

$$+ a^\dagger_n(\vec{k})\, e^{+i(\omega_{\vec{k}} + \omega_{\vec{p}})t} \underbrace{\int d^3r\, e^{-i(\vec{k}+\vec{p})\cdot\vec{r}}}_{V\delta_{\vec{k},-\vec{p}}} \Big)$$

⊕ Eq. (340.1):

$$\int d^3r\, e^{ip\cdot x}\, \hat{\epsilon}_s(\vec{p})\cdot\vec{A}(\vec{r},t) = \left(\frac{\hbar c^2 V}{2\omega_{\vec{p}}}\right)^{1/2} \left(a_s(\vec{p}) + (-1)^s a^\dagger_s(-\vec{p}) e^{2i\omega_{\vec{p}} t}\right)$$

(348.2)

similar:

$$\int d^3r\, e^{ip\cdot x}\, \hat{\epsilon}_s(\vec{p})\cdot\dot{\vec{A}}(\vec{r},t) = -i \left(\frac{\hbar c^2 \omega_{\vec{p}} V}{2}\right)^{1/2} \left(a_s(\vec{p}) - (-1)^s a^\dagger_s(-\vec{p}) e^{2i\omega_{\vec{p}} t}\right)$$

(348.3)

$$\hookrightarrow a_s(\vec{p}) = \frac{1}{(2\hbar c^2 V \omega_{\vec{p}})^{1/2}} \int d^3r\, e^{ip\cdot x}\, \hat{\epsilon}_s(\vec{p})\cdot\left(\omega_{\vec{p}}\vec{A}(x) + i\dot{\vec{A}}(x)\right)$$

e $\quad a^{\dagger}_n(\vec{k}) = \dfrac{1}{(2\hbar c^2 V \omega_{\vec{k}})^{1/2}} \displaystyle\int d^3 r' \, e^{-ikx'} \hat{\epsilon}_n(\vec{k}) \cdot \left(\omega_{\vec{k}} \vec{A}(x') - i \dot{\vec{A}}(x') \right)$

(349.1)

· como $\pi^k = \dfrac{1}{c} \partial^k \Phi - \dfrac{1}{c^2} A^k = -\dfrac{1}{c} \Lambda^k$: gauge de Coulomb

↳ Eq. (345.2): $[\dot{A}^i(\vec{r},t), A^j(\vec{r}',t)] = -i\hbar c^2 \delta^{Tr}_{ij}(\vec{r}-\vec{r}')$

(349.2)

· Dessa forma:

$$[a_s(\vec{p}), a^{\dagger}_n(\vec{k})] = \frac{1}{(2\hbar c^2 V \omega_{\vec{k}})^{1/2}} \frac{1}{(2\hbar c^2 V \omega_{\vec{p}})^{1/2}} \int d^3 r' d^3 r \, e^{ipx - ikx'} \, *$$

$$* \underbrace{[\omega_{\vec{p}} \epsilon^i_s(\vec{p}) A^i(\vec{r},t) + i \epsilon^i_s(\vec{p}) \dot{A}^i(\vec{r},t) ; \omega_{\vec{k}} \epsilon^j_n(\vec{k}) A^j(\vec{r}',t) - i \epsilon^j_n(\vec{k}) \dot{A}^j(\vec{r}',t)]}$$

$$= -i\omega_{\vec{p}} \epsilon^i_s(\vec{p}) \epsilon^j_n(\vec{k}) \underbrace{[A^i(\vec{r},t) ; \dot{A}^j(\vec{r}',t)]}_{+i\hbar c^2 \delta^{Tr}_{ij}(\vec{r}-\vec{r}')} +$$

$$+ i\omega_{\vec{k}} \epsilon^i_s(\vec{p}) \epsilon^j_n(\vec{k}) \underbrace{[\dot{A}^i(\vec{r},t) ; A^j(\vec{r}',t)]}_{-i\hbar c^2 \delta^{Tr}_{ij}(\vec{r}-\vec{r}')}$$

$$= \frac{\hbar(\omega_{\vec{k}} + \omega_{\vec{p}}) c^2}{2\hbar c^2 V (\omega_{\vec{k}} \omega_{\vec{p}})^{1/2}} e^{i(\omega_{\vec{p}} - \omega_{\vec{k}})t} \sum_{ij} \epsilon^i_s(\vec{p}) \epsilon^j_n(\vec{k}) \, *$$

$$* \underbrace{\int d^3 r \, d^3 r' \, e^{-i(\vec{p}\cdot\vec{r} - \vec{k}\cdot\vec{r}')} \delta^{Tr}_{ij}(\vec{r}-\vec{r}')}_{(I)}$$

⊕ Eq. (345.1):

$$(I) = \frac{1}{V}\sum_{\vec{q}}\int d^3r\, d^3r'\, e^{-i\vec{q}\cdot(\vec{r}-\vec{r}')}\, e^{+i\vec{k}\cdot(\vec{r}-\vec{r}')}\left(\delta_{ij} - \frac{q_i q_j}{q^2}\right)$$

$$= V\,\delta_{\vec{p},\vec{k}}\left(\delta_{ij} - \frac{p_i p_j}{p^2}\right)$$

$$\hookrightarrow \sum_{ij} \epsilon^i_s(\vec{p})\,\epsilon^j_n(\vec{k}) + (I) = V\,\delta_{\vec{k},\vec{p}}\Big(\underbrace{\hat{\epsilon}_s(\vec{p})\cdot\hat{\epsilon}_n(\vec{p})}_{\delta_{n,s}} -$$

$$- \frac{1}{p^2}\,\underbrace{\vec{p}\cdot\hat{\epsilon}_s(\vec{p})}_{0}\;\underbrace{\vec{p}\cdot\hat{\epsilon}_n(\vec{p})}_{0}\Big)$$

$$\hookrightarrow [a_s(\vec{p}), a^\dagger_n(\vec{k})] = \delta_{n,s}\,\delta_{\vec{k},\vec{p}}$$

· sobre hamiltoniano (347.1).

Eq. (346.2):

$$\vec{\nabla}\times\vec{A}(\vec{r},t) = +i\sum_{n=1,2}\sum_{\vec{k}}\left(\frac{\hbar c^2}{2V\omega_{\vec{k}}}\right)^{1/2}\vec{k}\times\hat{\epsilon}_n(\vec{k})\left(a_n(\vec{k})e^{-ik\cdot x} - a^\dagger_n(\vec{k})e^{+ik\cdot x}\right) \quad (350.1)$$

Eqs. (347.1), (348.1) e (350.1):

$$H = \frac{1}{2}\int d^3r \,:\frac{1}{c^2}\dot{A}^2 + (\vec{\nabla}\times\vec{A})^2:$$

$$= \frac{-1}{2}\sum_{n,s}\sum_{\vec{k},\vec{p}}\left(\frac{\hbar c^2}{2V\omega_{\vec{k}}}\right)^{1/2}\left(\frac{\hbar c^2}{2V\omega_{\vec{p}}}\right)^{1/2} *$$

$$*\underbrace{\left(\frac{1}{c^2}\hat{\epsilon}_n(\vec{k})\cdot\hat{\epsilon}_s(\vec{p})\,\omega_{\vec{k}}\,\omega_{\vec{p}} + (\vec{k}\times\epsilon_n(\vec{k}))\cdot(\vec{p}\times\epsilon_s(\vec{p})\right)}_{(I)} *$$

$$*\Big(a_n(\vec{k})\,a_s(\vec{p})\,e^{-i(\omega_{\vec{k}}+\omega_{\vec{p}})t}\underbrace{\int d^3r\, e^{i(\vec{k}+\vec{p})\cdot\vec{r}}}_{V\delta_{\vec{k},-\vec{p}}}$$

$$+ a^{\dagger}_n(\vec{k})\,a^{\dagger}_s(\vec{p})\,e^{+i(\omega_{\vec{k}}+\omega_{\vec{p}})t}\underbrace{\int d^3r\, e^{-i(\vec{k}+\vec{p})\cdot\vec{r}}}_{V\delta_{\vec{k},-\vec{p}}}$$

$$- a_n(\vec{k})\,a^{\dagger}_s(\vec{p})\,e^{-i(\omega_{\vec{k}}-\omega_{\vec{p}})t}\underbrace{\int d^3r\, e^{i(\vec{k}-\vec{p})\cdot\vec{r}}}_{V\delta_{\vec{k},\vec{p}}}$$

$$- a^{\dagger}_n(\vec{k})\,a_s(\vec{p})\,e^{+i(\omega_{\vec{k}}-\omega_{\vec{p}})t}\underbrace{\int d^3r\, e^{-i(\vec{k}-\vec{p})\cdot\vec{r}}}_{V\delta_{\vec{k},\vec{p}}}\Big)$$

como

(i) $\vec{k}=\vec{p}$: $(I) = \frac{1}{c^2}\omega^2_{\vec{k}}\underbrace{\hat{\epsilon}_s(\vec{k})\cdot\hat{\epsilon}_n(\vec{k})}_{\delta_{n,s}} + \underbrace{(\vec{k}\times\epsilon_n(\vec{k}))\cdot(\vec{k}\times\epsilon_s(\vec{k}))}_{k^2\epsilon_n(\vec{k})\cdot\epsilon_s(\vec{k}) - \vec{k}\cdot\epsilon_n(\vec{k})\,\vec{k}\cdot\epsilon_s(\vec{k})}$

$$k^2\underbrace{\epsilon_n(\vec{k})\cdot\epsilon_s(\vec{k})}_{\delta_{n,s}} - \underbrace{\vec{k}\cdot\epsilon_n(\vec{k})}_{0}\,\underbrace{\vec{k}\cdot\epsilon_s(\vec{k})}_{0}$$

$\hookrightarrow (I) = \left(\frac{1}{c^2}\omega^2_{\vec{k}} + k^2\right)\delta_{n,s} = \frac{2}{c^2}\omega^2_{\vec{k}}\,\delta_{n,s}$.

(ii) $\vec{k}=-\vec{p}$: $(I) = \left(\frac{1}{c^2}\omega^2_{\vec{k}} - k^2\right)\delta_{n,s} = 0$

· sobre o spin do fóton.

considerar: $|\psi\rangle = a^{\dagger}_{\lambda}(\vec{k})|0\rangle$: estado 1 - fóton

em particular $\vec{k} \parallel \hat{z}$;

ideia: calcular projeção componente spin $\parallel$ direção movimento, i.e., direção $\hat{k}$;

Eq. (342.2) : $\pi^i = -\frac{1}{c^2}\dot{A}^i$

⊕ Eq. (343.4) :

$$c^2 M^{ij} = \int d^3r \, :\dot{A}_i A^j - \dot{A}_j A^i: \; = -\int d^3r \, :\dot{A}^i A^j - \dot{A}^j \dot{A}:$$

notar: Eqs. (346.2) e (348.1),

$$\int d^3r \, \dot{A}^i A^j = (-i) \sum_{r,s} \sum_{\vec{k},\vec{p}} \left(\frac{\hbar c^2}{2V\omega_{\vec{k}}}\right)^{1/2} \left(\frac{\hbar c^2}{2V\omega_{\vec{q}}}\right)^{1/2} \omega_{\vec{k}} * \Big($$

$$\underbrace{\epsilon^i_r(\vec{k})\epsilon^j_s(\vec{q})}_{\epsilon^i_r(\vec{k})\epsilon^j_s(-\vec{k})} a_r(\vec{k}) a_s(\vec{q}) e^{-i(\omega_{\vec{k}}+\omega_{\vec{q}})t} \underbrace{\int d^3r \, e^{i(\vec{k}+\vec{q})\cdot\vec{r}}}_{V\delta_{\vec{k},-\vec{q}}}$$

$$-\underbrace{\epsilon^i_r(\vec{k})\epsilon^j_s(\vec{q})}_{\epsilon^i_r(\vec{k})\epsilon^j_s(-\vec{k})} a^{\dagger}_r(\vec{k}) a^{\dagger}_s(\vec{q}) e^{+i(\omega_{\vec{k}}+\omega_{\vec{q}})t} \underbrace{\int d^3r \, e^{-i(\vec{k}+\vec{q})\cdot\vec{r}}}_{V\delta_{\vec{k},-\vec{q}}}$$

$$+\underbrace{\epsilon^i_r(\vec{k})\epsilon^j_s(\vec{q})}_{\epsilon^i_r(\vec{k})\epsilon^j_s(\vec{k})} a_r(\vec{k}) a^{\dagger}_s(\vec{q}) e^{-i(\omega_{\vec{k}}-\omega_{\vec{q}})t} \underbrace{\int d^3r \, e^{i(\vec{k}-\vec{q})\cdot\vec{r}}}_{V\delta_{\vec{k},\vec{q}}}$$

$$- \underbrace{\epsilon^i_n(\vec{k})\, \epsilon^j_s(\vec{q})}_{\epsilon^i_n(\vec{k})\, \epsilon^j_s(\vec{k})}\, a^\dagger_n(\vec{k})\, a_s(\vec{q})\, e^{i(\omega_{\vec{k}} - \omega_{\vec{q}})t} \underbrace{\int d^3r\, e^{-i(\vec{k}-\vec{q})\cdot\vec{r}}}_{V\,\delta_{\vec{k},\vec{q}}}$$

$$\hookrightarrow \int d^3r\, \dot{A}^i A^j = \frac{-i}{2} \sum_{n,s} \sum_{\vec{k}} \hbar c^2 * \Big($$

$$\epsilon^i_n(\vec{k})\, \epsilon^j_s(-\vec{k}) \left(a_n(\vec{k})\, a_s(-\vec{k})\, e^{-2i\omega_{\vec{k}} t} - a^\dagger_n(\vec{k})\, a^\dagger_s(-\vec{k})\, e^{+2i\omega_{\vec{k}} t} \right)$$

$$+ \epsilon^i_n(\vec{k})\, \epsilon^j_s(\vec{k}) \left(a_n(\vec{k})\, a^\dagger_s(\vec{k}) - a^\dagger_n(\vec{k})\, a_s(\vec{k}) \right)$$

Dessa forma:

$$M^{ij} = i\frac{\hbar}{2} \sum_{n,s} \sum_{\vec{q}} :\epsilon^i_n(\vec{q})\, \epsilon^j_s(\vec{q}) * \Big(a_n(\vec{q})\, a^\dagger_s(\vec{k}) - a_s(\vec{q})\, a^\dagger_n(\vec{q})$$

$$- a^\dagger_n(\vec{q})\, a_s(\vec{q}) + a^\dagger_s(\vec{q})\, a_n(\vec{q}) \Big):$$

(353.1)

$$\hookrightarrow M^{ij} = i\hbar \sum_{n,s} \sum_{\vec{q}} \epsilon^i_n(\vec{q})\, \epsilon^j_s(\vec{q}) \left(a^\dagger_s(\vec{q})\, a_n(\vec{q}) - a^\dagger_n(\vec{q})\, a_s(\vec{q}) \right)$$

em particular,

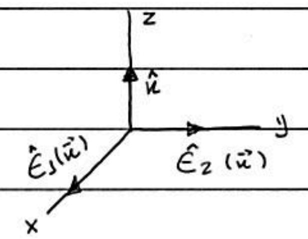
$$M^{12} |\psi\rangle = M^{12} a^\dagger_\lambda(\vec{k}) |0\rangle = [M^{12}, a^\dagger_\lambda(\vec{k})] |0\rangle + a^\dagger_\lambda(\vec{k}) \underbrace{M^{12} |0\rangle}_{=0}$$

como

$$[M^{12}, a^\dagger_\lambda(\vec{k})] = i\hbar \sum_s \epsilon^1_\lambda(\vec{k})\, \epsilon^2_s(\vec{k})\, a^\dagger_s(\vec{k}) -$$

$$- \epsilon^1_s(\vec{k})\, \epsilon^2_\lambda(\vec{k})\, a^\dagger_s(\vec{k})$$

como $\vec{k} \parallel \hat{z}$, temos que, veja pg. 340:



$\to \epsilon_1^1(\vec{k}) = 1$ e $\epsilon_1^2(\vec{k}) = 0$

$\epsilon_1^2(\vec{k}) = 0$ e $\epsilon_2^2(\vec{k}) = 0$

$\hookrightarrow [M^{12}, a_1^\dagger(\vec{k})] = i\hbar a_2^\dagger(\vec{k})$ e $[M^{12}, a_2^\dagger(\vec{k})] = -i\hbar a_1^\dagger(\vec{k})$, i.e.

estado $a_1^\dagger(\vec{k})|0\rangle$, foton c/ polarização linear, não é autoestado de M^{12};

entretanto, considerando

$$a_R^\dagger(\vec{k}) = \frac{1}{\sqrt{2}}\left(a_1^\dagger(\vec{k}) + i a_2^\dagger(\vec{k})\right)$$: polarização circular direita

e $$a_L^\dagger(\vec{k}) = \frac{1}{\sqrt{2}}\left(a_1^\dagger(k) - i a_2^\dagger(\vec{k})\right)$$: " " esquerda

$\hookrightarrow [M^{12}, a_R^\dagger(\vec{k})] = +\hbar a_R^\dagger(\vec{k})$

e $[M^{12}, a_L^\dagger(\vec{k})] = -\hbar a_L^\dagger(\vec{k})$

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