# FI 002 – Mecânica Quântica II – Lista 6

P.01. P.3.1, Mandl and Shaw:
From the expansion (3.7) for the real Klein-Gordon field $\phi(x)$, derive the following expression for the absorption operator $a(\mathbf{k})$:

$$a(\mathbf{k}) = \frac{1}{\sqrt{2\hbar c^2 V \omega_{\mathbf{k}}}} \int d^3 r e^{ikx} \left( i\dot{\phi}(x) + \omega_{\mathbf{k}} \phi(x) \right).$$

Hence derive the commutation relations (3.9) for the creation and annihilation operators from the commutation relations (3.6) for the fields.

P.02. P.13.2 and 13.7, Schwabl**:
For the real Klein-Gordon field $\varphi(x)$, show that the momentum operator

$$\mathbf{P} = -\int d^3 r : \pi(x) \nabla \varphi(x) :$$

assumes the form

$$\mathbf{P} = \sum_{\mathbf{k}} \hbar \mathbf{k} a^\dagger(\mathbf{k}) a(\mathbf{k}).$$

Moreover, show that the components of $P^\mu$ satisfies the commutator

$$[P^\mu, \varphi(x)] = -i\hbar \partial^\mu \varphi(x).$$

P.03. P.13.12 and 13.19, Schwabl:
For the Dirac field $\psi(x)$, show that the momentum operator

$$\mathbf{P} = -\int d^3 r : \pi_\alpha(x) \nabla \psi_\alpha(x) : \; = -i\hbar \int d^3 r : \psi^\dagger(x) \nabla \psi(x) :$$

assumes the form

$$\mathbf{P} = \sum_{r=1,2} \sum_{\mathbf{p}} \mathbf{p} \left( c_r^\dagger(\mathbf{p}) c_r(\mathbf{p}) + d_r^\dagger(\mathbf{p}) d_r(\mathbf{p}) \right).$$

Moreover, show that the components of $P^\mu$ satisfies the commutator

$$[P^\mu, \psi(x)] = -i \partial^\mu \psi(x).$$

P.04. P.14.2, Schwabl:
For the electromagnetic field $A^\mu(x)$, show that the momentum operator in the Coulomb gauge

$$\mathbf{P} = \frac{1}{c}\int d^3r : \mathbf{E}\times\mathbf{B} := -\frac{1}{c^2}\int d^3r : \dot{A}^i(x)\nabla A^i(x) :$$

assumes the form

$$\mathbf{P} = \sum_{r=1,2}\sum_{\mathbf{k}} \hbar\mathbf{k} a_r^\dagger(\mathbf{k}) a_r(\mathbf{k}).$$

P.05. P.13.15, Schwabl:
For the Dirac field $\psi(x)$, show that the charge operator

$$Q = q\int d^3r : \psi^\dagger(x)\psi(x) :$$

assumes the form

$$Q = -e\sum_{r=1,2}\sum_{\mathbf{p}} \left(c_r^\dagger(\mathbf{p})c_r(\mathbf{p}) - d_r^\dagger(\mathbf{p})d_r(\mathbf{p})\right),$$

where $q = -e < 0$. Moreover, show that $Q$ satisfies

$$[Q, c_r^\dagger(\mathbf{p})] = -ec_r^\dagger(\mathbf{p}) \quad \text{and} \quad [Q, d_r^\dagger(\mathbf{p})] = +ed_r^\dagger(\mathbf{p}).$$

06. P.2.3, Mandl and Shaw:
Show that the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\partial_\alpha\phi_\beta(x)\partial^\alpha\phi^\beta(x) + \frac{1}{2}\partial_\alpha\phi^\alpha(x)\partial_\beta\phi^\beta(x) + \frac{1}{2}\mu^2\phi_\alpha(x)\phi^\alpha(x)$$

for the real vector field $\phi^\alpha(x)$ leads to the field equations

$$\left[g_{\alpha\beta}(\Box + \mu^2) - \partial_\alpha\partial_\beta\right]\phi^\beta(x) = 0,$$

and that the field $\phi^\alpha(x)$ satisfies the Lorentz condition $\partial_\alpha\phi^\alpha(x) = 0$.

07. P.2.4, Mandl and Shaw:
Use the commutation relations (2.31)

$$[\phi_r(\mathbf{r},t), \pi_s(\mathbf{r}',t)] = i\hbar\delta(\mathbf{r}-\mathbf{r}'), \quad [\phi_r(\mathbf{r},t), \phi_s(\mathbf{r}',t)] = [\pi_r(\mathbf{r},t), \pi_s(\mathbf{r}',t)] = 0,$$

to show that the momentum operator of the fields

$$P^j = \int d^3r\, \pi_r(x)\frac{\partial\phi_r(x)}{\partial x_j}$$

satisfies the equations

$$\left[P^j, \phi_r(x)\right] = -i\hbar\frac{\partial\phi_r(x)}{\partial x_j}, \qquad \left[P^j, \pi_r(x)\right] = -i\hbar\frac{\partial\pi_r(x)}{\partial x_j}.$$

08. P.3.6, Mandl and Shaw:
The parity transformation (i.e. space inversion) of the Hermitian Klein-Gordon field $\phi(x)$ is defined by

$$\phi(\mathbf{r},t) \to \mathcal{P}\phi(\mathbf{r},t)\mathcal{P}^{-1} = \eta_p \phi(-\mathbf{r},t) \tag{1}$$

where the parity operator $\mathcal{P}$ is a unitary operator which leaves the vacuum invariant, $\mathcal{P}|0\rangle = |0\rangle$ and $\eta_p = \pm 1$ is called the intrinsic parity of the field. Show that the parity transformation leaves the Lagrangian density (3.4),

$$\mathcal{L} = \frac{1}{2}\left(\partial_\alpha \phi(x)\partial^\alpha \phi(x) - \mu^2 \phi(x)\phi(x)\right),$$

invariant. Show that $\mathcal{P}_1 a(\mathbf{k})\mathcal{P}_1^{-1} = ia(\mathbf{k})$ and $\mathcal{P}_2 a(\mathbf{k})\mathcal{P}_2^{-1} = -i\eta_p a(-\mathbf{k})$, where the $a(\mathbf{k})$ are the annihilation operators of the field, and $\mathcal{P}_1$ and $\mathcal{P}_2$ are given by

$$\mathcal{P}_1 = \exp\left[-i\frac{\pi}{2}\sum_{\mathbf{k}} a^\dagger(\mathbf{k})a(\mathbf{k})\right], \qquad \mathcal{P}_2 = \exp\left[+i\frac{\pi}{2}\eta_p \sum_{\mathbf{k}} a^\dagger(\mathbf{k})a(\mathbf{k})\right],$$

Hence, show that the operator $\mathcal{P} = \mathcal{P}_1\mathcal{P}_2$ is unitary and satisfies Eq. (1), i.e. it gives an explicit expression for the parity operator $\mathcal{P}$.

09. P.4.5, Mandl and Shaw:
For a Dirac field, the transformations

$$\psi(x) \to \psi'(x) = \exp(i\alpha\gamma_5)\,\psi(x), \qquad \psi^\dagger(x) \to \psi'^\dagger(x) = \psi(x)^\dagger \exp(-i\alpha\gamma_5),$$

where $\alpha$ is an arbitrary real parameter, are called chiral phase transformations. Show that the Lagrangian density (4.20),

$$\mathcal{L} = c\bar{\psi}(x)\left(i\hbar\gamma^\mu\partial_\mu - mc\right)\psi(x),$$

is invariant under chiral phase transformations in the zero-mass limit $m = 0$ only, and that the corresponding conserved current in this limit is the axial vector current $J_A^\mu(x) = \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$.
Deduce the equations of motion for the fields

$$\psi_L(x) = \frac{1}{2}(1-\gamma_5)\psi(x), \qquad \psi_R(x) = \frac{1}{2}(1+\gamma_5)\psi(x),$$

for non-vanishing mass, and show that they decouple in the limit $m = 0$. Hence show that the Lagrangian density

$$\mathcal{L} = i\hbar c\bar{\psi}_L\gamma^\mu\partial_\mu\psi_L(x)$$

describes zero-mass fermions with negative helicity only, and zero-mass antifermions with positive helicity only. (This field is called the Weyl field and can be used to describe the neutrinos in weak interactions in the approximation of zero mass.)