FI 193 – Teoria Quântica de Sistemas de Muitos Corpos – Lista 3

P.01. P.3.3, Miranda and P.x.x, Cologne: The noninteracting Anderson model.
The Hamiltonian of the noninteracting Anderson model (resonant level model) for spinless electrons is given by

$$H = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}} + E_f f^{\dagger} f + \sum_{\mathbf{k}} \big(V^{*}_{\mathbf{k}} f^{\dagger} c_{\mathbf{k}} + V^{*}_{\mathbf{k}} c^{\dagger}_{\mathbf{k}} f \big), \tag{1}$$

where $f^{\dagger}$ creates a localized electron at the impurity site $\mathbf{r} = 0$ and $c^{\dagger}_{\mathbf{k}}$ creates a conduction electron with momentum $\mathbf{k}$. Recall that the Green's function operator is defined as

$$(\Omega - H)\hat{G} = \mathbf{1}, \qquad \text{with} \qquad \Omega = \omega - i\eta. \tag{2}$$

(a) Consider the operator (2) and the single-particle states $|\mathbf{k}\rangle = c^{\dagger}_{\mathbf{k}}|0\rangle$ and $|f\rangle = f^{\dagger}|0\rangle$, where $|0\rangle$ is the vacuum state, and derive the following set of equations for the impurity $G_{f,f}(\Omega)$ and the conduction electron $G_{\mathbf{k},\mathbf{p}}(\Omega)$ Green's functions:

$$(\Omega - E_f)G_{f,f}(\Omega) = 1 + \sum_{\mathbf{k}} V^{*}_{\mathbf{k}} G_{\mathbf{k},f}(\Omega), \tag{3}$$

$$(\Omega - \epsilon_{\mathbf{k}})G_{\mathbf{k},f}(\Omega) = V_{\mathbf{k}} G_{f,f}(\Omega), \tag{4}$$

$$(\Omega - \epsilon_{\mathbf{k}})G_{\mathbf{k},\mathbf{p}}(\Omega) = \delta_{\mathbf{k},\mathbf{p}} + V_{\mathbf{k}} G_{f,\mathbf{p}}(\Omega), \tag{5}$$

$$(\Omega - E_f)G_{f,\mathbf{p}}(\Omega) = \sum_{\mathbf{k}} V^{*}_{\mathbf{k}} G_{\mathbf{k},\mathbf{p}}(\Omega). \tag{6}$$

Here $G_{\alpha,\beta}(\Omega) = \langle\alpha|\hat{G}(\Omega)|\beta\rangle = \langle 0|c_{\alpha}\hat{G}(\Omega)c^{\dagger}_{\beta}|0\rangle$, with $|\alpha\rangle$, $|\beta\rangle = |\mathbf{k}\rangle$ and $|f\rangle$. Notice that the impurity breaks translation invariance, and therefore, momentum is no longer a good quantum number.

(b) Solve the set of equations (3)–(6), and determine the impurity $G_{f,f}(\Omega)$ and the conduction electron $G_{\mathbf{k},\mathbf{p}}(\Omega)$ Green's functions.

(c) The density of states at the impurity level is given by

$$\rho_f(\omega) = \sum_{n} \delta(\omega - E_n)|\langle n|f\rangle|^2,$$

where $|n\rangle$ are eigenstates of the total Hamiltonian (1), i.e., $H|n\rangle = E_n|n\rangle$. Show that $\rho_f(\omega)$ can be written as

$$\rho_f(\omega) = \frac{1}{\pi}\mathrm{Im}\left[G_{f,f}(\omega - i\eta)\right] = -\frac{1}{\pi}\mathrm{Im}\left[G_{f,f}(\omega + i\eta)\right].$$

(d) Assume that the (hybridization) potential $V_{\mathbf{k}} = V$ is constant and that the density of states of the conduction electrons is constant,

$$\rho_c(\omega) = \sum_{\mathbf{k}} \delta\left(\omega - \epsilon_{\mathbf{k}}\right) = \rho_0, \qquad -D < \omega < D.$$

Show that the density of states of the impurity level $\rho_f(\omega)$ is approximately a Lorentzian centered around E_f with a width $\Gamma = \pi\rho_0 V^2$. Consider that $|E_f|$ and $\Gamma \ll D$.

(e) Show that the conduction electron Green's function $G_{\mathbf{k},\mathbf{p}}(\Omega)$ can be written as

$$G_{\mathbf{k},\mathbf{p}}(\Omega) = \delta_{\mathbf{k},\mathbf{p}} G^0_{\mathbf{k},\mathbf{k}}(\Omega) + G^0_{\mathbf{k},\mathbf{k}}(\Omega) T_{\mathbf{k},\mathbf{p}}(\Omega) G^0_{\mathbf{k},\mathbf{k}}(\Omega)$$

or, in the operator form,

$$\hat{G}(\Omega) = \hat{G}_0(\Omega) + \hat{G}_0(\Omega)\hat{T}(\Omega)\hat{G}_0(\Omega), \tag{7}$$

and determine the matrix element $T_{\mathbf{k},\mathbf{p}}(\Omega) = \langle \mathbf{k}|T|\mathbf{p}\rangle$ of the so-called $\hat{T}$ matrix in terms of the (hybridization) potential $V_{\mathbf{k}}$. Here $G^0_{\mathbf{k},\mathbf{k}}(\Omega) = (\Omega - \epsilon_{\mathbf{k}})^{-1}$ indicates the conduction electron Green's function in the absence of impurity.

(f) In scattering theory, the S-matrix is defined in terms of the phase shit $\delta(\omega)$ and the T-matrix is related to the S-matrix [see Eq.(20.12), Merzbacher]:

$$S(\omega) = e^{2i\delta(\omega)} \qquad \text{and} \qquad S(\omega) = 1 - 2\pi i \rho(\omega) T(\omega + i\eta),$$

where $\rho(\omega)$ is the density of states. Consider the conduction electrons and the assumptions of item (d) and show that the phase shift

$$\delta(\omega) = \tan^{-1}\left(\frac{\Gamma}{E_f - \omega}\right),$$

which indicates a scattering resonance, see Sec.13.6 from Merzbacher for details.

02. P.4.1, Fetter and Walecka: Hartree-Fock approximation.
A uniform spin-S Fermi system has a spin-independent interaction potential $V(\mathbf{r}) = (V_0/r)e^{-r/a}$, where V_0 and a are constants.

(a) Evaluate the proper self-energy in the Hartree-Fock approximation. Hence find the excitation spectrum $\epsilon_{\mathbf{k}}$ and the Fermi energy $\epsilon_F = \mu$.

(b) Show that the exchange contribution to the Fermi energy ϵ_F is negligible for a long-range interaction ($k_F a \gg 1$), but that the direct and exchange terms are comparable for a short-range interaction ($k_F a \ll 1$).

(c) In this approximation, prove that the effective mass m^* is determined solely by the exchange contribution. Compute m^*, and discuss the limiting cases $k_F a \gg 1$ and $k_F a \ll 1$.

(d) What is the relation between the limit $a \to \infty$ of this model and the electron gas in a uniform positive background?

P.03. P.4.1 and P.4.2, Bruus and Flensberg: Hartree-Fock approximation.
A system of interacting fermions is described by the Hamiltonian

$$H = H_0 + V = \sum_{\alpha} \epsilon_\alpha c_\alpha^\dagger c_\alpha + \frac{1}{2} \sum_{\alpha\beta\mu\nu} V_{\alpha\beta;\nu\mu} c_\alpha^\dagger c_\beta^\dagger c_\mu c_\nu. \tag{8}$$

Consider that the expectation values $\bar{n}_{\alpha\beta} = \langle c_\alpha^\dagger c_\beta \rangle \neq 0$ and that $c_\alpha^\dagger c_\beta \approx \langle c_\alpha^\dagger c_\beta \rangle$, i.e., the deviations $\left(c_\alpha^\dagger c_\nu - \langle c_\alpha^\dagger c_\nu \rangle\right) \equiv d_{\alpha\nu}$ can be seen as small parameters. In this case, we consider the expectation values $\bar{n}_{\alpha\beta}$ as mean-field parameters. Importantly, such a choice is not unique, it should be based on physical arguments, and it depends on the system and/or phase to be described.

(a) Assuming that $\beta \neq \nu$, we have $c_\alpha^\dagger c_\beta^\dagger c_\mu c_\nu = c_\alpha^\dagger c_\nu c_\beta^\dagger c_\mu$. The operator $c_\alpha^\dagger c_\nu$ can be written in terms of the deviation $d_{\alpha\nu}$ as

$$c_\alpha^\dagger c_\nu = \langle c_\alpha^\dagger c_\nu \rangle + \left(c_\alpha^\dagger c_\nu - \langle c_\alpha^\dagger c_\nu \rangle \right). \tag{9}$$

Write the operator $c_\beta^\dagger c_\mu$ as in Eq. (9), and show that the interaction term V is approximately given by

$$V^{\text{Hartree}} = \frac{1}{2} \sum_{\alpha\beta\mu\nu} V_{\alpha\beta;\nu\mu} \left(\bar{n}_{\beta\mu} c_\alpha^\dagger c_\nu + \bar{n}_{\alpha\nu} c_\beta^\dagger c_\mu - \bar{n}_{\alpha\nu} \bar{n}_{\beta\mu} \right). \tag{10}$$

Such a term is known as the Hartree approximation for the interaction term.

(b) Assume that $\beta \neq \mu$, and write the interaction term V in terms of the operators $c_\alpha^\dagger c_\mu$ and $c_\beta^\dagger c_\nu$. Then, show that the interaction term V can be approximately written as

$$V^{\text{Fock}} = -\frac{1}{2} \sum_{\alpha\beta\mu\nu} V_{\alpha\beta;\nu\mu} \left(\bar{n}_{\alpha\mu} c_\beta^\dagger c_\nu + \bar{n}_{\beta\nu} c_\alpha^\dagger c_\mu - \bar{n}_{\alpha\mu} \bar{n}_{\beta\nu} \right), \tag{11}$$

which is known as the Fock term. Notice that, within the approximations (10) and (11), the Hamiltonian (8) assumes the form

$$H^{\text{HF}} = H_0 + V^{\text{Hartree}} + V^{\text{Fock}}, \tag{12}$$

i.e., within the Hartree-Fock (mean-field) approximation, the interacting problem (8) is reduced to a single-particle problem. In principle, the mean-field Hamiltonian (12) can be diagonalized and the single-particle energy levels can be determined, once the mean-field parameters $\bar{n}_{\alpha\beta}$ are self-consistently determined.

(c) Consider now the homogeneous electron gas [see Eq. (3.25), Fetter]. Due to translation invariance, the expectation value $\langle c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{q}\alpha} \rangle$ is diagonal, i.e., the mean-field parameters $n_{\mathbf{k}\,\alpha} = \delta_{\mathbf{k},\mathbf{q}} \langle c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{q}\alpha} \rangle$. In this case, show that, in principle, the Hartree-Fock Hamiltonian (12) assumes the form

$$H^{\text{HF}} = E_0 + \sum_{\mathbf{k}\alpha} \epsilon_{\mathbf{k}\,\alpha}^{\text{HF}} c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{k}\alpha}, \tag{13}$$

where E_0 is a constant to be determined and

$$\epsilon^{\mathrm{HF}}_{\mathbf{k}\,\alpha} = \epsilon_{\mathbf{k}} + nU(0) - \frac{1}{V}\sum_{\mathbf{q}} U(\mathbf{p}-\mathbf{k}) n_{\mathbf{p}\,\alpha} = \epsilon_{\mathbf{k}} + nV(0) + V_{\mathrm{HF}}(k), \tag{14}$$

with the number of particles $N = \sum_{\mathbf{k},\alpha} n_{\mathbf{k}\,\alpha}$, the particle density $n = N/V$, and $U(\mathbf{q}) = 4\pi e^2/q^2$. Recall that, due to the positively charged background, the direct term $U(0)$ is absent and only the exchange term $V_{\mathrm{HF}}(k)$ is finite. Finally, determine the self-consistent equation that allow us to calculate the mean-field parameters $n_{\mathbf{k}\,\alpha}$.

(d) Consider the system at temperature $T = 0$. Assume that $n_{\mathbf{k}\alpha} = \Theta(k_F - k)$ and show that

$$V_{\mathrm{HF}}(k) = -\frac{e^2 k_F}{\pi}\left(1 + \frac{k_F^2 - k^2}{2k_F k}\ln\left|\frac{k+k_F}{k-k_F}\right|\right).$$

Plot $V_{\mathrm{HF}}(k)$ in terms of k/k_F and, based on the behaviour of $V_{\mathrm{HF}}(k)$, argue that the choice $n_{\mathbf{k}\alpha} = \Theta(k_F - k)$ is indeed the correct solution of the self-consistent equation determined in item (c).

(e) Determine the ground state energy E^0_{HF} within the Hartree-Fock approximation (it is not necessary to evaluate the momentum integrals). Compare the obtained result with Eqs. (3.30) and (3.31) from Fetter and notice that E^0_{HF} is equal to the ground state energy of the electron gas determined within first-order perturbation theory.

(f) Show that the density of states determined from Eq. (14) diverges at the Fermi level. Such a result contradicts both experiments and the Fermi liquid theory. It also warns us that the single-particle energies derived form a mean-field Hamiltonian are not necessary a good approximation for the excitation energies of the system, even if the mean-field approach gives a good estimate for the ground state energy.

P.04. P.x.x, Bruus and Flensberg: The Stoner model of metallic ferromagnets.
Transition metals, where the conduction bands are formed by narrow d or f orbitals, may display metallic magnetism. The interaction between two particles in these orbitals is stronger than the one between electrons occupying the more spread s or p orbitals, and therefore, correlation effects are stronger in the former than in the latter. Typical metals, where correlations between conduction band electrons are important, are Fe and Ni. Due to the fact that the short range part of the interaction is the most important ingredient, such (itinerant) metallic systems can be described by the Hubbard model, whose Hamiltonian in momentum space is given by

$$H = \sum_{\mathbf{k}\alpha} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\,\alpha} c_{\mathbf{k}\,\alpha} + \frac{1}{2V} U \sum_{\mathbf{k}\,\mathbf{q}\,\mathbf{p}} \sum_{\alpha\beta} c^\dagger_{\mathbf{k}+\mathbf{q}\,\alpha} c^\dagger_{\mathbf{p}-\mathbf{q}\,\beta} c_{\mathbf{p}\,\beta} c_{\mathbf{k}\,\alpha}, \tag{15}$$

where $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu = \hbar^2 k^2/2m - \mu$, $U > 0$, and V is the system volume. In the following, we consider the Hubbard model within the Hartree-Fock approximation.

(a) Consider the spin-dependent mean-field parameters $\langle c^\dagger_{\mathbf{k}\,\alpha} c_{\mathbf{q}\,\alpha}\rangle = \delta_{\mathbf{k},\mathbf{q}}\bar{n}_{\mathbf{k}\,\alpha}$, and show that, within the Hartree-Fock approximation, the Hamiltonian (15) assumes the form

$$H^{\rm HF} = \sum_{\mathbf{k}\alpha} \xi^{\rm HF}_{\mathbf{k},\alpha} c^\dagger_{\mathbf{k}\,\alpha} c_{\mathbf{k}\,\alpha} - \frac{1}{2V} U \sum_{\alpha\beta} \bar{n}_\alpha \bar{n}_\beta + \frac{1}{2V} U \sum_{\alpha} \bar{n}^2_\alpha,$$

where

$$\xi^{\rm HF}_{\mathbf{k},\alpha} = \epsilon_{\mathbf{k}} - \mu + U\left(\bar{n}_\uparrow + \bar{n}_\downarrow - \bar{n}_\alpha\right) \qquad \text{and} \qquad \bar{n}_\alpha = \frac{1}{V}\sum_{\mathbf{k}} \langle c^\dagger_{\mathbf{k}\,\alpha} c_{\mathbf{k}\,\alpha}\rangle.$$

(b) The self-consistent equation that allows us to determine the mean-field parameters $\bar{n}_\alpha$ is given by

$$\bar{n}_\alpha = \frac{1}{V}\sum_{\mathbf{k}} \langle c^\dagger_{\mathbf{k}\,\alpha} c_{\mathbf{k}\,\alpha}\rangle = \frac{1}{V}\sum_{\mathbf{k}} n_{\rm FD}\left(\xi^{\rm HF}_{\mathbf{k},\alpha}\right), \tag{16}$$

where $n_{\rm FD}(x) = 1/[\exp(\beta x) + 1]$ is the Fermi-Dirac function. Consider the self-consistent condition (16) at temperature $T = 0$ and determine the relation between the Fermi wavevector $k_{F,\alpha}$ and $\bar{n}_\alpha$, for $\alpha =\uparrow$ and $\downarrow$. Then, using the fact that $\xi^{\rm HF}_{k_F,\alpha} = 0$, show that

$$\frac{\hbar^2}{2m}\left(6\pi^2\bar{n}_\uparrow\right)^{2/3} + U\bar{n}_\downarrow = \mu \qquad \text{and} \qquad \frac{\hbar^2}{2m}\left(6\pi^2\bar{n}_\downarrow\right)^{2/3} + U\bar{n}_\uparrow = \mu,$$

which are indeed the self-consistent equations to be solved.

(c) Define the variables

$$\lambda = \frac{\bar{n}_\uparrow - \bar{n}_\downarrow}{\bar{n}}, \qquad \gamma = \frac{2mU}{\hbar^2}\left(\frac{1}{3\pi^2}\right)^{2/3}\bar{n}^{1/3}, \qquad \text{where} \qquad \bar{n} = \bar{n}_\uparrow + \bar{n}_\downarrow,$$

and show that

$$\gamma\lambda = (1+\lambda)^{2/3} - (1+\lambda)^{2/3} \equiv f(\lambda). \tag{17}$$

Plot $f(\lambda)$ and $f'(0)\lambda$ for $0 \le \lambda \le 1$, and determine the values of γ which corresponds to
(i) the normal state phase ($\lambda = 0$),
(ii) the weak ferromagnet phase ($0 < \lambda < 1$), and
(iii) the strong ferromagnet phase ($\lambda = 1$).
Obs.: The three possible solutions of Eq. (17) are illustrated in Fig. 4.5, Bruus.

(d) At $T = 0$, show that the critical Hubbard coupling U_c above with a (strong) ferromagnetic phase sets in is given by

$$\frac{U_c}{3V}\rho(0) = 1,$$

where $\rho(0)$ is the density of states at the Fermi energy.
Obs.: See Fig. 13.1, Coleman for the behaviour of U_c in terms of the temperature.

P.05. P.x.x, Bruus and Flensberg: The random phase approximation (RPA).
Consider the interacting electron gas (spinless fermions) whose Hamiltonian is given by

$$H = H_0 + V_{\rm int} = \sum_{\mathbf{k}} \xi_{\mathbf{k}} c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}} + \frac{1}{2V} \sum_{\mathbf{k}\,\mathbf{p}\,\mathbf{q}\neq 0} U(q) c_{\mathbf{k}+\mathbf{q}}^{\dagger} c_{\mathbf{p}-\mathbf{q}}^{\dagger} c_{\mathbf{p}} c_{\mathbf{k}}, \tag{18}$$

where $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu = \hbar^2 k^2/2m - \mu$ and the $\mathbf{q} = 0$ component is excluded from the interaction term due to the presence of the positively charged background. The idea is to derive the retarded density-density correlation function in the RPA for the interacting electron gas within the equation of motion technique. Recall that the retarded density-density correlation function is defined as ($\hbar = 1$)

$$\chi^R(\mathbf{q}, t - t') = -i\theta(t - t') \frac{1}{V} \langle \Big[\rho(\mathbf{q}, t),\, \rho(-\mathbf{q}, t') \Big] \rangle \qquad \text{where} \qquad \rho(\mathbf{q}) = \sum_{\mathbf{k}} c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}+\mathbf{q}}.$$

In particular, for the free-electron gas (spinless fermions), we have

$$\chi_0^R(\mathbf{q}, \nu) = \frac{1}{V} \sum_{\mathbf{k}} \frac{n_{\mathbf{k}} - n_{\mathbf{k}+\mathbf{q}}}{\nu + \xi_{\mathbf{k}} - \xi_{\mathbf{k}+\mathbf{q}} + i\eta},$$

where $\langle c_{\mathbf{k}}^{\dagger} c_{\mathbf{q}} \rangle = \delta_{\mathbf{k},\mathbf{q}} n_{\mathbf{k}}$.

(a) It is interesting to consider the auxiliary correlation function

$$\chi^R(\mathbf{k}, \mathbf{q}, t - t') = -i\theta(t - t') \langle \Big[\bar{\rho}(\mathbf{k}, \mathbf{q}, t),\, \rho(-\mathbf{q}, t') \Big] \rangle.$$

where

$$\bar{\rho}(\mathbf{k}, \mathbf{q}, t) = e^{iHt} \bar{\rho}(\mathbf{k}, \mathbf{q}) e^{-iHt} = e^{iHt} c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}+\mathbf{q}} e^{-iHt}.$$

Note that $\chi^R(\mathbf{q}, t - t') = (1/V) \sum_{\mathbf{k}} \chi^R(\mathbf{k}, \mathbf{q}, t - t')$. Show that the equation of motion for the correlation function $\chi^R(\mathbf{k}, \mathbf{q}, t - t')$ is given by

$$\begin{aligned} i\partial_t \chi^R(\mathbf{k}, \mathbf{q}, t - t') &= \delta(t - t') \langle [\bar{\rho}(\mathbf{k}, \mathbf{q}, t),\, \rho(-\mathbf{q}, t')] \rangle \\ &- i\theta(t - t') \langle \Big[[H,\, \bar{\rho}(\mathbf{k}, \mathbf{q}, t)]\,,\, \rho(-\mathbf{q}, t') \Big] \rangle. \end{aligned}$$

(b) Determine the commutators

$$[\bar{\rho}(\mathbf{k}, \mathbf{q}), \rho(\mathbf{q})]\,, \qquad [H_0,\, \bar{\rho}(\mathbf{k}, \mathbf{q})]\,, \qquad [V_{\rm int},\, \bar{\rho}(\mathbf{k}, \mathbf{q})]\,.$$

(c) Consider the commutator $[V_{\rm int},\, \bar{\rho}(\mathbf{k}, \mathbf{q})]$ within a mean-field approximation: Replace the operators $c_{\mathbf{k}}^{\dagger} c_{\mathbf{q}}$ by its expectation values $\langle c_{\mathbf{k}}^{\dagger} c_{\mathbf{q}} \rangle = \delta_{\mathbf{k},\mathbf{q}} n_{\mathbf{k}}$; consider only the direct terms (Hartree approximation) as done in item (a) from Problem 03; then, show that

$$\Big[V_{\rm int},\, c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}+\mathbf{q}} \Big] \approx \frac{1}{V} U(q) \left(n_{\mathbf{k}+\mathbf{q}} - n_{\mathbf{k}} \right) \sum_{\mathbf{p}} c_{\mathbf{p}-\mathbf{q}}^{\dagger} c_{\mathbf{p}}.$$

(d) Consider the equation of motion for the correlation function $\chi^R(\mathbf{k}, \mathbf{q}, t - t')$ in the frequency domain and show that

$$\chi^R_{\mathrm{RPA}}(\mathbf{q}, \nu) = \frac{\chi^R_0(\mathbf{q}, \nu)}{1 - U(q)\chi^R_0(\mathbf{q}, \nu)}.$$

P.06. P.6.4 and P.6.5, Bruus and Flensberg: conductivity homogenous system.
In this problem, we consider the conductivity of a translation- and rotational-invariant system. In this case, the conductivity $\sigma(\mathbf{r}, \mathbf{r}') = \sigma(\mathbf{r} - \mathbf{r}')$ and the conductivity tensor is diagonal with identical diagonal components (see Sec. 6.2, Bruus for details).

(a) Show that in the Fourier domain, Eq. (6.15) from Bruus assumes the form

$$\mathbf{J}_e(\mathbf{q}, \omega) = \sigma(\mathbf{q}, \omega)\mathbf{E}(\mathbf{q}, \omega).$$

(b) Find the relation between the conductivity $\sigma(\mathbf{q}, \omega)$ and the correlation function

$$C_\alpha(\mathbf{q}, t) = \langle [J^\alpha(\mathbf{q}, t),\, J^\alpha(-\mathbf{q}, t)] \rangle,$$

where $\mathbf{J}(\mathbf{q})$ is the particle current operator in momentum space.

(c) Consider the noninteracting electron gas in the long-wavelength limit $\mathbf{q} \to 0$. Derive the expression for the particle current operator in this limit,

$$\mathbf{J}(0, t) = \frac{1}{m} \sum_{\mathbf{k},\alpha} \mathbf{k} c^\dagger_{\mathbf{k}\,\alpha}(t) c_{\mathbf{k}\,\alpha}(t),$$

and show that it is time-independent in the Heisenberg picture. Then, show that the conductivity

$$\sigma^{\alpha,\beta}(\mathbf{q} \to 0, \omega) = i\delta_{\alpha,\beta} \frac{ne^2}{\omega m}.$$

How does this fit with the Drude result [Eq. (13.42), Bruus] in the clean limit, where the impurity induced scattering time τ is such that $\omega\tau \to \infty$?

(d) How does the conclusions from the previous item change for an interacting translation-invariant system?