FI 193 – Teoria Quântica de Sistemas de Muitos Corpos – Lista 4

P.01. Matsubara frequency summation.

(a) Show that

$$
\begin{aligned}
S_1 = \frac{1}{\beta\hbar}\sum_n \mathcal{G}^{(0)}(\mathbf{p}, i\omega_n)\mathcal{G}^{(0)}(\mathbf{p}+\mathbf{q}, i\omega_n + i\nu_m) &= \frac{f(\xi_{\mathbf{p}}) - f(\xi_{\mathbf{p}+\mathbf{q}})}{i\nu_m + \xi_{\mathbf{p}}/\hbar - \xi_{\mathbf{p}+\mathbf{q}}/\hbar}, \\
S_2 = -\frac{1}{\beta\hbar}\sum_n \mathcal{G}^{(0)}(\mathbf{p}, i\omega_n)\mathcal{G}^{(0)}(\mathbf{q}-\mathbf{p}, i\nu_m - i\omega_n) &= \frac{1 - f(\xi_{\mathbf{p}}) - f(\xi_{\mathbf{q}-\mathbf{p}})}{i\nu_m - \xi_{\mathbf{p}}/\hbar - \xi_{\mathbf{q}-\mathbf{p}}/\hbar}, \\
S_3 = -\frac{1}{\beta\hbar}\sum_m \mathcal{D}^{(0)}(\mathbf{q}, i\nu_m)\mathcal{G}^{(0)}(\mathbf{p}+\mathbf{q}, i\omega_n + i\nu_m) &= \frac{n(\omega_{\mathbf{q}}) + f(\xi_{\mathbf{p}+\mathbf{q}})}{i\omega_n + \omega_{\mathbf{q}} - \xi_{\mathbf{p}+\mathbf{q}}/\hbar} \\
&+ \frac{1 + n(\omega_{\mathbf{q}}) - f(\xi_{\mathbf{p}+\mathbf{q}})}{i\omega_m - \omega_{\mathbf{q}} - \xi_{\mathbf{p}+\mathbf{q}}/\hbar},
\end{aligned}
$$

where $\nu_m = 2m\pi/\beta\hbar$, $\mathcal{G}^{(0)}(\mathbf{p}, i\omega_n) = (i\omega_n - \xi_{\mathbf{p}}/\hbar)^{-1}$ is the Matsubara Green's function for free fermions, $\mathcal{D}^{(0)}(\mathbf{p}, i\omega_n)$ is the Matsubara Green's function for free phonons, $\hbar\omega_{\mathbf{q}}$ is the phonon energy, $n(x)$ is the Bose-Einstein function, and $f(x)$ is the Fermi-Dirac function.

(b) Express the Matsubara Green's functions for fermions $\mathcal{G}(\mathbf{p}, i\omega_n)$ and for phonons $\mathcal{D}(\mathbf{q}, i\nu_n)$ in terms of the respectives spectral functions, i.e,

$$
\mathcal{G}(\mathbf{p}, i\omega_n) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\frac{A(\mathbf{p},\omega)}{i\omega_n - \omega} \qquad \text{and} \qquad \mathcal{D}(\mathbf{q}, i\nu_m) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\frac{B(\mathbf{q},\omega')}{i\nu_m - \omega'},
$$

and determine the Matsubara frequency summation

$$
S_4 = -\frac{1}{\beta\hbar}\sum_m \mathcal{D}(\mathbf{q}, i\nu_m)\mathcal{G}(\mathbf{p}+\mathbf{q}, i\omega_n + i\nu_m).
$$

P.02. P.5.2, Miranda: Electron-phonon interaction.
Consider a system of free fermions interacting with the (bosonic) lattice vibrations (phonons). The Hamiltonian of the system is given by

$$H = H_e + H_{\text{ph}} + H_{\text{e-ph}}, \tag{1}$$

where

$$H_e = \sum_{\mathbf{k}\,\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\,\sigma} c_{\mathbf{k}\,\sigma}, \qquad \xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu = \hbar^2 k^2/(2m) - \mu, \tag{2}$$

$$H_{\text{ph}} = \sum_{\mathbf{q}} \omega_{\mathbf{q}} a^\dagger_{\mathbf{q}} a_{\mathbf{q}}, \qquad \omega_{\mathbf{q}} = cq < cq_0 = \omega_D, \tag{3}$$

$$H_{\text{e-ph}} = \frac{1}{V^{1/2}} \sum_{\mathbf{k}\,\mathbf{q}} M_{\mathbf{q}} c^\dagger_{\mathbf{k}+\mathbf{q}} c_{\mathbf{k}} \left(a^\dagger_{-\mathbf{q}} + a_{\mathbf{q}} \right), \qquad M_{\mathbf{q}} = g_{\mathbf{q}} \sqrt{q} = M^*_{-\mathbf{q}}. \tag{4}$$

Here $c^\dagger_{\mathbf{k}\,\sigma}$ ($c_{\mathbf{k}\,\sigma}$) creates (annihilates) an electron with momentum $\mathbf{k}$ and spin σ, $a^\dagger_{\mathbf{q}}$ and $a_{\mathbf{q}}$ are boson operators associated with acoustic phonons, $\omega_D = cq_0 \ll E_F$ is the Debye frequency with $E_F = \hbar^2 k_F^2/2m$ being the Fermi energy and q_0 being a momentum cutoff, and μ is the chemical potential for the electrons (recall that, for phonons, $\mu = 0$ since they are not conserved). Let us assume that the electron-phonon coupling $M_{\mathbf{q}} = g\sqrt{q}$, with g being a constant.

(a) The Matsubara Green's function for phonons is defined as

$$\mathcal{D}(\mathbf{q}, \tau) = -\left\langle T_\tau \left[\left(a_{-\mathbf{q}}(\tau) + a^\dagger_{\mathbf{q}}(\tau) \right) \left(a_{\mathbf{q}}(0) + a^\dagger_{-\mathbf{q}}(0) \right) \right] \right\rangle.$$

Determine the Matsubara Green's function $\mathcal{D}^{(0)}(\mathbf{q}, \nu_n)$ for the system of free phonons. Notice that $\mathcal{D}^{(0)}(\mathbf{q}, \nu_n) \propto \theta(q_0 - q)$.

(b) Determine the self-energy $\Sigma^{(2)}_\sigma(\mathbf{k}, \omega_n)$ for the electrons in second-order perturbation theory, i.e., up to order g^2.

(c) Performe an analytic continuation on $\Sigma^{(2)}_\sigma(\mathbf{k}, \omega_n)$ and show that the advanced self-energy for the electrons is given by

$$\Sigma^A(\mathbf{k}, \omega) = \int \frac{d^3p}{(2\pi)^3} \lambda_{\mathbf{p}} \left[\frac{1 + n(\omega_{\mathbf{p}}) - f(\xi_{\mathbf{k}-\mathbf{p}})}{\omega - \xi_{\mathbf{k}-\mathbf{p}} - \omega_{\mathbf{p}} - i\eta} + \frac{n(\omega_{\mathbf{p}}) + f(\xi_{\mathbf{k}-\mathbf{p}})}{\omega - \xi_{\mathbf{k}-\mathbf{p}} + \omega_{\mathbf{p}} - i\eta} \right], \tag{5}$$

where $\lambda_{\mathbf{p}} = |M_{\mathbf{p}}|^2 \theta(q_0 - p)$ and $n(x)$ and $f(x)$ are the Bose-Einstein and Fermi-Dirac distribution functions, respectively.

(d) From now on, consider the $T = 0$ limit. Assume that $q_0 = k_F$ (since they are of the same order) and show that

$$\left| \frac{1}{v_F} \frac{\partial}{\partial k} \mathrm{Re}\left[\Sigma^A(\mathbf{k}, 0)\right]\Big|_{k=k_F} \right| \ll \left| \frac{\partial}{\partial \omega} \mathrm{Re}\left[\Sigma^A(k_F, \omega)\right]\Big|_{\omega=0} \right|, \tag{6}$$

where $v_F = k_F/m \gg c$. Condition (6) indicates that the dependence of the self-energy with momentum can be neglected.
Hint: Use the fact that $(\mathbf{k} - \mathbf{p})^2 = k^2 + p^2 - 2kp\mu$, where $\mu = \cos\theta$.

(e) From

$$\frac{\partial}{\partial \omega} \mathrm{Re}\left[\Sigma^A(k_F, \omega)\right]\Big|_{\omega=0},$$

determine the renormalized Fermi velocity and the quasiparticule residue Z due to the electron-phonon interaction. See Sec. 15.4, Bruus for details.

(f) Determine the quasiparticle decay rate $\mathrm{Im}\left[\Sigma^{(2A)}_\sigma(k_F, \omega)\right]$ as a function of the frequency for $0 < \omega < \omega_D$. Show that it obeys the condition $\mathrm{Im}\left[\Sigma^{(2A)}_\sigma(k_F, \omega)\right] \ll \omega$ when $\omega \to 0$ (recall Fermi liquid theory).

(g) Consider the so-called Migdal function

$$\alpha^2 F(\nu, \hat{\mathbf{k}}) \equiv \int \frac{dS_{\hat{\mathbf{p}}}}{(2\pi)^3 v_{\hat{\mathbf{p}}}} |M_{\mathbf{k}-\mathbf{p}}|^2 \theta\left(q_0 - |\mathbf{k} - \mathbf{p}|\right) \delta\left(\nu - \omega_{\mathbf{k}-\mathbf{p}}\right),$$

where $dS_{\hat{\mathbf{p}}}$ is a area element of the Fermi surface in the $\hat{\mathbf{p}}$ direction, $v_{\hat{\mathbf{p}}} \nabla \xi_{\mathbf{p}}$ is the Fermi velocity in the $\hat{\mathbf{p}}$ direction, and the momenta $\mathbf{k}$ and $\mathbf{p}$ are on the Fermi surface. Show that (5) is approximately given by

$$\Sigma^A(k_F, \omega) = \int_0^\infty d\nu \int_{-\infty}^{\infty} d\xi \alpha^2 F(\nu, \hat{\mathbf{k}}) \left[\frac{1 + n(\nu) - f(\xi)}{\omega - \xi - \nu - i\eta} + \frac{n(\nu) + f(\xi)}{\omega - \xi + \nu - i\eta} \right].$$

Consider the $T = 0$ limit and show that

$$\begin{aligned}
\Sigma^A(k_F, \omega) &= \int_0^\infty d\nu \alpha^2 F(\nu, \hat{\mathbf{k}}) \ln\left(\frac{\omega - \nu - i\eta}{\omega + \nu - i\eta}\right), \\
-\frac{\partial}{\partial \omega} \mathrm{Re}\left[\Sigma^A(k_F, \omega)\right]\Big|_{\omega=0} &= 2\int_0^\infty d\nu \frac{\alpha^2 F(\nu, \hat{\mathbf{k}})}{\nu}, \qquad 0 < \omega < \omega_D, \\
\mathrm{Im}\left[\Sigma^A(k_F, \omega)\right] &= \pi \int_0^\omega d\nu \alpha^2 F(\nu, \hat{\mathbf{k}}).
\end{aligned}$$

Hint: For the integral over ξ, assume that $-D < \xi < D$, where ω, $\nu \ll D$.

(h) Since $dS_{\hat{\mathbf{p}}} = k_F^2 d\Omega_{\hat{\mathbf{p}}}$ and $v_{\hat{\mathbf{p}}} = v_F$, show that

$$\alpha^2 F(\nu) = \frac{\rho_F g^2}{2c}\left(\frac{\nu}{\omega_D}\right)^2 \theta(\nu)\theta(\omega_D - \nu), \tag{7}$$

where $\rho_F = mk_F^2/(2\pi^2)$ is the electron density (per spin) on the Fermi surface. Rederive the results of itens (e) and (f) from Eq. (7).

(i) Consider the self-energy (5) at finite temperatures and the Migdal function (7). Determine $\mathrm{Im}\left[\Sigma^A(k_F, 0)\right]$ as a function of temperature for $T \ll \omega_D$.

03. P.7.3, Fetter and Walecka:
Assuming a uniform system of spin-$1/2$ fermions at temperature T, and using the Feynman rules in momentum space:

(a) Write out the second-order contributions to the proper self-energy in the case of a spin-independent interaction;

(b) Evaluate the frequency sums.

04. P.7.4, Fetter and Walecka: Free fermions on a external potential.
Consider a system of noninteracting particles in an external static potential with a Hamiltonian

$$H^{ex} = \int d^3r\, \psi^\dagger_\alpha V_{\alpha\beta}(\mathbf{r})\psi_\beta(\mathbf{r}).$$

(a) Use Wick's theorem to evaluate the temperature Green's to second order in H^{ex}. Hence, deduce the Feynman rules for $\mathcal{G}^{ex}_{\alpha\beta}(\mathbf{r}\,\tau, \mathbf{r}'\,\tau')$ to all orders.

(b) Define the Fourier transform

$$\mathcal{G}^{ex}(\mathbf{r}\,\tau, \mathbf{r}'\,\tau') = \frac{1}{\beta\hbar}\sum_n \int \frac{d^3k\, d^3q}{(2\pi)^6} e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\mathbf{q}\cdot\mathbf{r}'} e^{-i\omega_n(\tau-\tau')}\mathcal{G}^{ex}_{\alpha\beta}(\mathbf{k},\mathbf{q};\omega_n).$$

Find $\mathcal{G}^{ex}_{\alpha\beta}(\mathbf{k},\mathbf{q};\omega_n)$ to second order, and hence obtain the corresponding Feynman rules in momentum space.

(c) Show that Dyson's equation becomes

$$\begin{aligned}\mathcal{G}^{ex}_{\alpha\beta}(\mathbf{k},\mathbf{q};\omega_n) \quad &= \quad (2\pi)^3\delta(\mathbf{k}-\mathbf{q})\mathcal{G}^0_{\alpha\beta}(\mathbf{k};\omega_n) \\ &+ \quad \frac{1}{(2\pi)^3\hbar}\int d^3p\, \mathcal{G}^0_{\alpha\lambda}(\mathbf{k};\omega_n)V_{\lambda\lambda'}(\mathbf{k}-\mathbf{p})\mathcal{G}^{ex}_{\lambda'\beta}(\mathbf{p},\mathbf{q};\omega_n).\end{aligned}$$

(d) Express the internal energy and thermodynamic potential in a form analogous to Eqs. (23.15) and (23.22) from Fetter and Walecka.

05. P.7.5, Fetter and Walecka: Spin$-1/2$ fermions under a magnetic field.
Apply the theory of Prob. 7.4, Fetter and Walecka, to a system of spin-$1/2$ fermions in a uniform magnetic field, where $V_{\alpha\beta} = -\mu_0 \mathbf{H} \cdot \boldsymbol{\sigma}_{\alpha\beta}$.

(a) Express the magnetization $\mathbf{M}$ (magnetic moment per unit volume) in terms of G^{ex} and $\mathcal{G}^{ex}$ respectively for $T = 0$ and $T \geq 0$.

(b) Solve Dyson's equation in each case and find $\mathbf{M}$; hence obtain the following limits

$$\chi_P = \frac{3\mu^2 n_0}{3\epsilon_F} \qquad \text{as } T \to 0 \qquad \text{Pauli paramagnetism,}$$

$$\chi_C = \frac{\mu_0^2 n}{K_B T} \qquad \text{as } T \to \infty \qquad \text{Curie's law,}$$

where n is the particle density.

(c) Why does the zero-temperature formalism give the wrong answer?

06. P.2.2, Mahan: Electron-phonon interaction.
For the phonon Green's fumction $D(\mathbf{q}, t - t')$, let $V(t)$ be the electron-phonon interaction and evaluate all the $n = 2$ diagrams. Which are connected, and which are disconnected? Draw the Feynman graphs for each term.

P.07. P.x.x, Cologne: Slave-boson approximation for the Kondo model.
The idea of this problem is to study the Kondo model within the so-called slave-boson approximation.
Let us consider the Kondo model

$$\hat{H} = \hat{H}_0 + \hat{H}_1 = \sum_{\mathbf{k}\,\alpha} \varepsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\,\alpha} c_{\mathbf{k}\,\alpha} + J\mathbf{S}\cdot\mathbf{s}_0, \tag{8}$$

where $c^{\dagger}_{\mathbf{k}\,\alpha}$ ($c_{\mathbf{k}\,\alpha}$) creates (annihilates) a conduction electron with momentum $\mathbf{k}$ and spin $\alpha =\uparrow,\downarrow$, $\varepsilon_{\mathbf{k}}$ is the fermion dispersion, J is the Kondo coupling, $\mathbf{S}$ is the impurity spin-$1/2$ operator, and $\mathbf{s}_0$ is the conduction electron spin operator at the impurity site, i.e.,

$$\mathbf{s}_0 = \frac{1}{2}\sum_{\mathbf{k},\mathbf{p}}\sum_{\alpha,\beta} c^{\dagger}_{\mathbf{k}\,\alpha}\hat{\tau}_{\alpha,\beta} c_{\mathbf{p}\beta},$$

with $\hat{\tau} = (\sigma_x, \sigma_y, \sigma_z)$ being the vector of Pauli matrices.

(a) It is convenient to write the impurity spin operator $\mathbf{S}$ in terms of auxiliary fermion operators f_σ (the so-called Abrikosov fermions), i.e., $\mathbf{S} = 1/2\sum_{\mu,\nu} f^{\dagger}_{\mu}\hat{\tau}_{\mu,\nu} f_{\nu}$. In order to preserve the size of the Hilbert space, we need a constraint, $\sum_\alpha f^{\dagger}_{\alpha} f_\alpha = 1$. Show that, apart from a constant, $\hat{H}_1$ assumes the form

$$\hat{H}_1 = -\sum_{\alpha,\beta}\frac{J}{2} f^{\dagger}_{\alpha} c_\alpha c^{\dagger}_{\beta} f_\beta, \tag{9}$$

where $c_\alpha \equiv c_\alpha(\mathbf{r}=0) = \sum_{\mathbf{k}} c_{\mathbf{k}\alpha}$ is the conduction electron operator at the impurity site.

(b) Consider the four-fermion term $(J/2)f^{\dagger}cc^{\dagger}f$ within a Hartree-Fock (mean field) approximation by replacing the *bosonic operator* $c^{\dagger}f$ by its average value. Introduce a Lagrange multiplier λ to enforce the occupation constraint on the impurity site and assume that λ is constant. Show that, apart from a constant, the Hamiltonian (8) assumes the form

$$H^{MF} = \sum_{\mathbf{k}\,\alpha} \varepsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\,\alpha} c_{\mathbf{k}\,\alpha} - \sum_{\alpha}\Big(b\, f^{\dagger}_{\alpha} c_\alpha + b^{\dagger}\, f_\alpha c^{\dagger}_{\alpha}\Big) + \lambda\left(\sum_\alpha f^{\dagger}_{\alpha} f_\alpha - 1\right), \tag{10}$$

with

$$b = \frac{J}{2}\sum_\sigma \langle c^{\dagger}_{\sigma} f_\sigma\rangle. \tag{11}$$

Notice that b is in general a complex number.

(c) The Hamiltonian (10) is bilinear in fermion operators and is thus solvable. It is useful to introduce a propagator $G_{\alpha,f}(\tau)$ for the f fermions and a mixed propagator $G_{\alpha,fc}(\tau)$, i.e.,

$$G_{\alpha,f}(\tau) = -\langle T_\tau f_\alpha(\tau) f_\alpha^\dagger(0)\rangle \quad \text{and} \quad G_{\alpha,fc}(\tau) = -\langle T_\tau f_\alpha(\tau) c_\alpha^\dagger(0)\rangle.$$

Show that

$$G_f(i\omega_n) \equiv G_{\alpha,f}(i\omega_n) = \frac{1}{i\omega_n - \lambda - |b|^2 G^0(i\omega_n)}, \tag{12}$$

$$G_{fc}(i\omega_n) \equiv G_{\alpha,fc}(i\omega_n) = -b\, G_f(i\omega_n) G^0(i\omega_n), \tag{13}$$

where $G^0(i\omega_n) \equiv G^0(\mathbf{r}=0, i\omega_n) = \sum_k (i\omega_n - \varepsilon_{\mathbf{k}})^{-1}$ is the local Green's function for the conduction electrons.
Hint: Recall the discussion about the noninteracting Anderson model.

(d) From now on, let us assume that b is real. The assumption made in item (b), that λ is constant, implies that the constraint $\sum_\alpha f_\alpha^\dagger f_\alpha = 1$ is fulfilled only on average, i.e.,

$$\sum_\alpha \langle f_\alpha^\dagger f_\alpha\rangle = 1. \tag{14}$$

Rewrite Eqs. (11) and (14) in terms of the Green's functions (12) and (13). Notice that the two derived equations together with Eq. (12) form a set of self-consistent equations. Once the density of states of the conduction electrons $\rho_0(\omega)$ and J are known, the equations can be solved for a fixed temperature.

(e) Use the results of item (d), convert the Matsubara sums into integrals over real frequencies, and show that Eq. (11) can be written as

$$\frac{1}{J} = -\int_{-\infty}^{\infty} d\omega\, n_{FD}(\omega) \frac{\rho_0(\omega)}{\omega - \lambda} \left| 1 - \frac{b^2 G^0(\omega + i\eta)}{\omega - \lambda} \right|^{-2}, \tag{15}$$

for small $b \neq 0$ and

$$\frac{1}{J} = -\int_{-\infty}^{\infty} d\omega\, n_{FD}(\omega) \left(\frac{\rho_0(\omega)}{\omega - \lambda} + \mathrm{Re}\, G^0(\omega + i\eta)\delta(\omega - \lambda) \right), \tag{16}$$

for $b \to 0$. Here the spectral density (density of states) $\rho_0(\omega) = -\mathrm{Im} G^0(\omega + i\eta)/\pi$ and $n_{FD}(x) = 1/[\exp(\beta x) + 1]$ is the Fermi function.
Hint: The identity $(1/\beta\hbar)\sum_{i\omega_n} G(i\omega_n) = \int d\omega \rho(\omega) n_f(\omega)$, where $\rho(\omega)$ is the spectral density related to $G(\omega + i\eta)$, might be useful.

(f) Assume that $\rho_0(\omega) = \rho_0$ is constant for $-D < \omega < D$, where D is the bandwidth of the conduction electrons, and discuss the solutions of the mean field equations. Show that it is possible to derive the correct (one-loop) expression for the Kondo temperature T_K from these equations.
Obs.: The slave-boson approximation introduces an artificial phase transition at T_K.

08. P.x.x, Cologne: Fermionic Green's function for a chain.
Consider a one-dimensional fermionic system described by the Hamiltonian

$$H = H_0 + V = \sum_{i=1}^{N} \left[t(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) - \mu c_i^\dagger c_i \right] + \sum_{i=1}^{N} \Delta \left(c_i c_{i+1} + c_{i+1}^\dagger c_i^\dagger \right), \tag{17}$$

where $c_i^\dagger$ (c_i) creates (annihilates) an electron at site i of the chain, t is the nearest-neighbor hopping energy, and μ and Δ are constants.

(a) Calculate the Green's function $G_0(\mathbf{q}, i\omega_n)$ for the noninteracting system.
Hint: Perform a Fourier transform.

(b) Calculate now the Green's function for the interacting system with the help of Dyson's equation. Why does the self-energy only involve even powers of V?

09. P.7.1, Cologne: Specific heat of a d-wave BCS superconductor.
The electronic specific heat of a superconductor is given by

$$C_S = T\frac{\partial S}{\partial T} = \sum_{\mathbf{k}\,\sigma} E_{\mathbf{k}} \frac{\partial f_K}{\partial T}. \tag{18}$$

Here $f_{\mathbf{k}} = 1/(\exp(E_{\mathbf{k}}/T) + 1)$ is the Fermi-Dirac distribution function and the second equality follows from the fact that the entropy for a Fermi gas can be written as

$$S = -\sum_{\mathbf{k}\,\sigma} \left[(1 - f_{\mathbf{k}}) \ln(1 - f_{\mathbf{k}}) + f_{\mathbf{k}} \ln f_{\mathbf{k}} \right].$$

Let us consider a d-wave BCS theory in a 2D square lattice. In this case, the energy of the elementary excitations are given by $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}$, with $\xi_{\mathbf{k}} = -2t(\cos k_x + \cos k_y) - \mu$ and $\Delta_{\mathbf{k}} = 2\Delta_0(\cos k_x - \cos k_y)$. Show that Eq.(18) can be written as

$$C_S \sim \int_{-\infty}^{\infty} d\xi \int \frac{d\theta}{2\pi} \sqrt{\xi^2 + \Delta_{\mathbf{k}}^2} \frac{\partial}{\partial T} \left(\frac{1}{\exp(\sqrt{\xi^2 + \Delta_{\mathbf{k}}^2}) + 1} \right) \tag{19}$$

and that for the d-wave case $C_S \sim T^2$ in the limit $T \to 0$.
Hint: The important contributions come from the gapless region at the nodal points.
Obs. 1: Notice that, in the derivation of the second equality in Eq. (18), we neglected the fact that $\Delta = \Delta(T)$. This procedure is justified in the limit of very low-T because in this case the T-dependence of the gap provides subleading corrections to the specific heat. Recall that in the BCS theory $\Delta(T) - \Delta(T = 0) \sim T^2$
Obs. 2: Recall that for the s-wave case, $C_S \sim (T_c/T)^{5/2} e^{-\Delta(T=0)/T}$.