F 415 – Mecânica Geral II – Lista 1

01. P.8.4, Marion:
Perform an explicit calculation of the time average (i.e., the average over one complete period) of the potential energy for a particle moving in an elliptical orbit in a central inverse-square-law force field. Express the result in terms of the force constant of the field and the semimajor axis of the ellipse. Perform a similar calculation for the kinetic energy. Compare the results and thereby verify the virial theorem for this case.

02. P.8.6, Marion:
Two gravitating masses m_1 and m_2 $(m_1 + m_2 = M)$ are separated by a distance r_0 and released from rest. Show that when the separation is $r < r_0$, the speeds are

$$v_1 = m_2\sqrt{\frac{2G}{M}\left(\frac{1}{r} - \frac{1}{r_0}\right)}, \qquad v_2 = m_1\sqrt{\frac{2G}{M}\left(\frac{1}{r} - \frac{1}{r_0}\right)},$$

P.03. P.8.7, Marion and P.1.10, Fetter:
A particle of mass m moves under the influence of an attractive force given by $F(r) = -kr$.

a) Use the effective potential to show that all orbits are bound and that the energy E must exceed $E_{min} = \sqrt{kl^2/m}$.

b) Verify that the orbit is a closed ellipse with origin at the center, i.e., with the aid of Eq. (8.17), Marion, show that

$$\frac{\alpha^2}{r^2} = 1 + A\cos(2\theta),$$

where

$$A^2 = 1 - \frac{E_{min}^2}{E^2}, \qquad \alpha^2 = \frac{l^2}{mE}.$$

c) If the relation $E/E_{min} = \cosh\xi$ defines the quantity ξ, verify that the orbital parameters are given

$$a^2 = e^{\xi}\frac{l}{\sqrt{mk}}, \qquad b^2 = e^{-\xi}\frac{l}{\sqrt{mk}}, \qquad \epsilon^2 = 1 - e^{-2\xi}.$$

d) Show that the period $\tau = 2\pi\sqrt{m/k}$.

e) Calculate the time averages of the kinetic and potential energies and, in particular, show that $\langle U\rangle = ka^2/2$. Compare the results with the virial theorem.

04. P.8.8, Marion:
Investigate the motion of a particle repelled by a force center according to the law $F(r) = kr$. Show that the orbit can only be hyperbolic.

05. P.8.10, Marion:
Assume Earth's orbit to be circular and that the Sun's mass suddenly decreases by half. What orbit does Earth then have? Will Earth escape the solar system?

P.06. P.8.11, Marion and 3.13, Goldstein:
A particle moves under the influence of a central force given by $F(r) = -k/r^n$. Consider that the particle's orbit is circular and passes through the force center.

a) Show that, in this case, $n = 5$.

b) Show that for the orbit described the total energy of the particle is zero.

c) Find the period of the motion.

d) Find $\dot{x}$, $\dot{y}$, and v as a function of angle around the circle and show that all three quantities are infinite as the particle goes through the center of force.

07. P.3.14, Goldstein:

a) For circular and parabolic orbits in an attractive $1/r$ potential having the same angular momentum, show that the perihelion distance of the parabola is one-half the radius of the circle.

b) Prove that in the same central force as in part (a) the speed of a particle at any point in a parabolic orbit is $\sqrt{2}$ times the speed in a circular orbit passing through the same point.

P.08. P.8.13, Marion, P.3.51, Symon and P.3.21, Goldstein:
Consider a particle of mass μ moving under the central force

$$F(r) = -\frac{k}{r^2} - \frac{\lambda}{r^3}, \qquad k, \lambda > 0.$$

a) Use the effective potential and discuss the possible types of motion. Consider the three cases: $\mu\lambda < l^2$, $\mu\lambda = l^2$, and $\mu\lambda > l^2$.

b) Solve the radial equation and show that the orbit is given by (case $\mu\lambda < l^2$)

$$r = \frac{a(1-\epsilon^2)}{1 + \epsilon\cos(\beta\theta)}.$$

c) Show that the above orbit is a precessing ellipse. Determine the angular velocity of precession and state whether the precession is in the same or in the opposite direction of the orbital angular velocity.

P.09. P.3.47, Symon, P.3.19, Goldstein and P.8.44, Marion:
The Yukawa potential adds an exponential term to the long–range Coulomb potential, which decreases the range of it:

$$U(r) = -\frac{ke^{-r/a}}{r}, \qquad k, a > 0.$$

It has great usefulness in atomic and nuclear physics.

a) Determine the force and compare it with the inverse–square–law force.

b) Discuss the possible types of motion for a particle of mass m moving under $U(r)$. Compare them with the ones obtained for the inverse–square–law force.

c) Investigate the stability of circular orbits. Determine the angular momentum and energy in terms of the orbit radius ρ. Determine the frequency of the circular motion and the frequency of small radial oscillations around ρ.

d) Show that the nearly circular orbits are almost closed when ρ is very small.

10. P.8.15, Marion:
A particle of unit mass moves from infinity along a straight line that, if continued, would allow it to pass a distance $b\sqrt{2}$ from a point P. If the particle is attracted toward P with a force varying as k/r^5, and if the angular momentum about the point P is $\sqrt{k}/b$, show that the trajectory is given by $r = b\coth(\theta/\sqrt{2})$.

11. P.8.17, Marion:
A particle moves in an elliptical orbit in an inverse-square-law central-force field. If the ratio of the maximum angular velocity to the minimum angular velocity of the particle in its orbit is n, then show that the eccentricity of the orbit is

$$\epsilon = \frac{\sqrt{n}-1}{\sqrt{n}+1}.$$

P.12. P.3.50, Symon and P.8.22, Marion:
Consider a particle under the central force

$$F(r) = -\frac{k}{r^3}, \qquad k > 0.$$

a) Use the effective potential and discuss the possible types of motion. In each case, determine the possible values (ranges) of the total energy E and angular momentum l.

b) Solve the radial equation and show that the solutions have the form

$$\frac{1}{r} = Af\left[\beta(\theta - \theta_0)\right],$$

where $f(x) = 1, \exp(x), \cos(x), \cosh(x), \sinh(x)$.

c) For each of the above solutions, determine the values of l and E. Determine A and β in terms of l and E.

d) Sketch the typical orbits.

13. P.8.23, Marion:
An Earth satellite moves in an elliptical orbit with a period τ, eccentricity ϵ, and semimajor axis a. Show that the maximum radial velocity of the satellite is $2\pi a\epsilon/(\tau\sqrt{1-\epsilon^2})$.

14. P.8.31, Marion:
Consider a force law of the form

$$F(r) = -\frac{k}{r^2} - \frac{k'}{r^4}.$$

Show that if $\rho^2 k > k'$, then a particle can move in a stable circular orbit at $r = \rho$.

P.15. P.8.32 and P.8.36, Marion:
A particle moves in a central force field described by

$$F(r) = -\frac{k}{r^2}\exp\left(-\frac{r}{a}\right), \qquad k, a > 0.$$

a) Discuss by the method of the effective potential the qualitative nature of the orbits. Determine the possible values of the angular momentum and total energy.

b) Investigate the stability of circular orbits.

c) Show that if the orbit is nearly circular, the apsides will advance approximately by $\pi\rho/a$ per revolution, where ρ is the radius of the circular orbit.

P.16. P.8.33, Marion:
Consider a particle of mass m constrained to move on the surface of a paraboloid whose equation (in cylindrical coordinates) is $r^2 = 4az$. If the particle is subject to a gravitational force, show that the frequency of small oscillations about a circular orbit with radius $\rho = \sqrt{4az_0}$ is

$$\omega = \sqrt{\frac{2g}{a+z_0}}.$$

17. P.8.35, Marion:
An almost circular orbit (i.e., $\epsilon \ll 1$) can be considered to be a circular orbit to which a small perturbation has been applied. Then, the frequency of the radial motion is given by Equation 8.89. Consider a case in which the force law is $F(r) = -k/r^n$ (where n is an integer), and show that the apsidal angle is $\pi/\sqrt{3-n}$. Thus, show that a closed orbit generally results only for the harmonic oscillator force and the inverse-square-law force (if values of n equal to or smaller than -6 are excluded).

18. P.8.45, Marion:
A particle of mass m moves in a central force field that has a constant magnitude F_0, but always points toward the origin.

a) Find the angular velocity ω_ϕ required for the particle to move in a circular orbit of radius r_0.

b) Find the frequency ω_r of small radial oscillations about the circular orbit. Both answers should be in terms of F_0, m, and r_0.

19. P.3.43, Symon:
The potential energy for an isotropic harmonic oscillator is

$$V = \frac{1}{2}kr^2.$$

Plot the effective potential energy for the r-motion when a particle of mass m moves with this potential energy and with angular momentum L about the origin. Discuss the types of motion that are possible, giving as complete a description as is possible without carrying out the solution. Find the frequency of revolution for circular motion and the frequency of small radial oscillations about this circular motion. Hence describe the nature of the orbits which differ slightly from circular orbits.

20. P.3.46, Symon:
A particle of mass m moves under the action of a central force whose potential is

$$V(r) = kr^4, \qquad k > 0.$$

For what energy and angular momentum will the orbit be a circle of radius a about the origin? What is the period of this circular motion? If the particle is slightly disturbed from this circular motion, what will be the period of small radial oscillations about $r = a$?

21. P.3.20 Goldstein and P.3.49, Symon:
A uniform distribution of dust in the solar system adds to the gravitational attraction of the Sun on a planet an additional central force

$$F = -mCr,$$

where m is the mass of the planet, C is a constant proportional to the gravitational constant and the density of the dust, and $\mathbf{r}$ is the radius vector from the Sun to the planet (both considered as points). This additional force is very small compared to the direct Sun-planet gravitational force.

a) Calculate the period for a circular orbit of radius r_0 of the planet in this combined field.

b) Calculate the period of radial oscillations for slight disturbances from this circular orbit.

c) Show that nearly circular orbits can be approximated by a precessing eHipse and find the precession frequency. Is the precession in the same or opposite direction to the orbital angular velocity?

P.22. P.3.75, Symon:
A particle of charge q in a cylindrical magnetron moves in a uniform magnetic field $\mathbf{B} = B\hat{z}$ and an electric field, directed radially outward or inward from a central wire along the z-axis,

$$\mathbf{E} = \frac{a}{\rho}\hat{\rho},$$

where ρ is the distance from the z-axis and $\hat{\rho}$ is a unit vector directed radially outward from the z-axis. The constants a and B may be either positive or negative.

a) Set up the equations of motion in cylindrical coordinates.

b) Show that the quantity $m\rho^2\dot{\phi} + qB\rho^2/(2c) = K$ is a constant of the motion.

c) Using this result, give a qualitative discussion, based on the energy integral, of the types of motion that can occur. Consider all cases, including all values of a, B, K, and E.

d) Under what conditions can circular motion about the axis occur?

e) What is the frequency of small radial oscillations about this circular motion?