Dinâmica sistema de partículas.

Refs.: Secs. 9.1 - 9.8, Marion

Secs. 4.1 - 4.4 e 4.6, Symon.

ideia: descrever sistema $n > 1$ partículas.

quantidades úteis: centro de massa;
forças internas;
momento linear total;
" angular ";
energia total.

em seguida: descrever colisões elásticas e inelásticas
e " corpo rígido.

considerar: sistema n partículas, $\alpha, \beta = 1, 2, \ldots, n$;
REF. O: referencial inercial

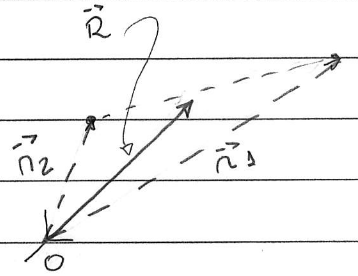
Definição: centro de massa (CM):

$$\vec{R} = \frac{1}{M} \sum_{\alpha=1}^{n} m_\alpha \vec{r}_\alpha, \tag{56.1}$$

onde:

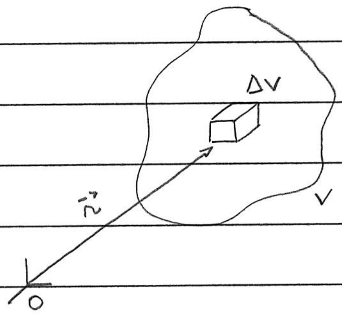
m_α: massa partícula α

$\vec{r}_\alpha$: posição " "
w.r.t. O



$$M = \sum_{\alpha=1}^{n} m_\alpha \; : \; \text{massa total} \tag{56.2}$$

p/ uma distribuição contínua de massa, temos que:



Δm_α : massa volume ΔV

(57.1)

Eq. (56.2) : $M = \sum_{\alpha=1}^{n} \Delta V \left(\frac{\Delta m_\alpha}{\Delta V} \right) \xrightarrow[\Delta V \to 0]{} \int d^3r \, \rho(\vec{r})$ (57.2)

onde $\rho(\vec{r}) = \lim_{\substack{\Delta V \to 0 \\ \vec{r} \in \Delta V}} \left(\frac{\Delta m}{\Delta V} \right)$: densidade (57.3)

↳ Eq. (56.1) : $\vec{R} = \frac{1}{M} \sum_{\alpha} \Delta V \left(\frac{\Delta m_\alpha}{\Delta V} \right) \vec{r}_\alpha$

$\xrightarrow[\Delta V \to 0]{} \frac{1}{M} \int d^3r \, \vec{r} \rho(\vec{r})$ (57.4)

Obs.: veja Exemplo 9.1, Marion e pg. 57.1 !

· próxima etapa: generalizar teoremas de conservação (discutidos no Cap. 2, Marion) p/ sistema n partículas;

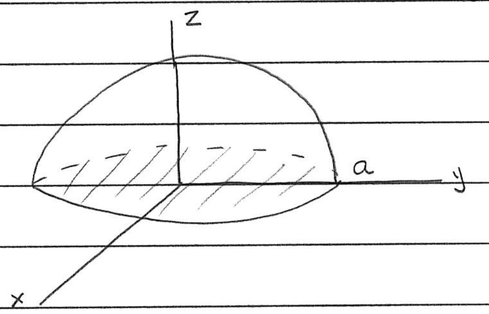
I) Conservação momento linear total,

Seja: $\vec{F}_\alpha = \vec{F}_\alpha^{(e)} + \vec{f}_\alpha$: força total sob partícula α (57.5)

$\vec{f}_\alpha$: resultante forças internas

$\vec{F}_\alpha^{(e)}$: " " externas

Ex. 1: Exemplo 9.1, Marion: considerar semi hemisfério raio a e densidade $\rho = cte$; determinar $\vec{R}$;



Eq. (57.1) ⊕ coordenadas esféricas:

$$\vec{R} = \frac{1}{M}\rho \int \vec{r}\, r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$= \frac{1}{M}\rho \int_0^a dr\, r^2 \int_0^{\pi/2} d\theta \sin\theta \int_0^{2\pi} d\varphi \; *$$

$$* \; r\left(\sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}\right)$$

$$= \frac{1}{M}\rho \underbrace{\int_0^a dr\, r^3}_{a^4/4} \underbrace{\int_0^{\pi/2} d\theta \sin\theta\cos\theta}_{\underbrace{\frac{1}{2}\sin 2\theta}_{-\frac{1}{4}\cos 2\theta\big|_0^{\pi/2} = \frac{1}{2}}} \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\, \hat{z} = \frac{1}{M}\rho\cdot\frac{\pi a^4}{4}$$

Eq. (57.2): $M = \rho \int_0^a dr\, r^2 \int_0^{\pi/2} d\theta \sin\theta \int_0^{2\pi} d\varphi = \rho\cdot\frac{2}{3}\pi a^3$

↳ $\vec{R} = \dfrac{3}{2\pi a^3}\cdot\dfrac{\pi a^4}{4}\,\hat{z} = \dfrac{3}{8}\,a\,\hat{z}$

de fato,

$$\vec{f}_\alpha = \sum_{\beta \neq \alpha} \vec{f}_{\alpha\beta} \qquad (58.1)$$

↳ força sob partícula α devido β

sobre as forças internas $\vec{f}_{\alpha\beta}$:

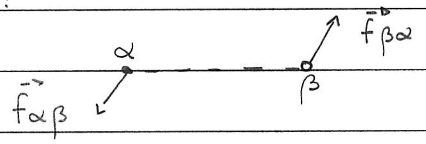
ok c/ 3ª lei de Newton, entretanto, duas possibilidades:

(i) $\vec{f}_{\alpha\beta} = -\vec{f}_{\alpha\beta}$: forma "fraca" 3ª lei

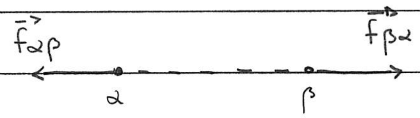
(58.2)

(ii) " ⊕ $\vec{f}_{\alpha\beta} \parallel \vec{r}_\alpha - \vec{r}_\beta$: " "forte" " .

notar:



: ok c/ (i)



: " " (ii)

· 2ª lei de Newton p/ partícula α: Eq. (58.1)

$$\frac{d\vec{p}_\alpha}{dt} = m_\alpha \frac{d^2\vec{r}_\alpha}{dt^2} = \vec{F}_\alpha = \vec{F}_\alpha^{(e)} + \vec{f}_\alpha = \vec{F}_\alpha^{(e)} + \sum_{\beta \neq \alpha} \vec{f}_{\alpha\beta} \qquad (58.3)$$

Eq. (57.1)

↳ p/ sistema n partículas, temos que:

$$\sum_{\alpha=1}^{n} m_\alpha \frac{d^2\vec{r}_\alpha}{dt^2} = \underbrace{\sum_\alpha \vec{F}_\alpha^{(e)}}_{\equiv \vec{F}} + \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \vec{f}_{\alpha\beta} \qquad (58.4)$$

$\equiv \vec{F}$: resultante forças externas notar: $\sum_{\alpha \neq \beta} \vec{f}_{\alpha\beta} = \frac{1}{2} \sum_{\alpha \neq \beta} \vec{f}_{\alpha\beta} + \underbrace{\vec{f}_{\alpha\beta}}_{\alpha \leftrightarrow \beta} = \frac{1}{2} \sum_{\alpha \neq \beta} \vec{f}_{\alpha\beta} + \vec{f}_{\beta\alpha}$

forma "fraca"
Eq. (58.2) $\;= \frac{1}{2} \sum_{\alpha \neq \beta} \vec{f}_{\alpha\beta} - \vec{f}_{\alpha\beta} = 0$

como: $\frac{d^2\vec{R}}{dt^2} = \frac{1}{M} \sum_{\alpha} m_\alpha \frac{d^2\vec{r}_\alpha}{dt^2}$: Eq. (56.1)

↳ Eq. (58.4) : $M \frac{d^2\vec{R}}{dt^2} = \vec{F}$ (59.1)

como: $\vec{p}_\alpha = m_\alpha \frac{d\vec{r}_\alpha}{dt}$: momento linear partícula α

↳ $\sum_\alpha \vec{p}_\alpha = \sum_\alpha m_\alpha \frac{d\vec{r}_\alpha}{dt} = \frac{d}{dt}\left(\sum_\alpha m_\alpha \vec{r}_\alpha\right) = \frac{d}{dt}(M\vec{R}) = \vec{P}$

↳ $\vec{P} = M \frac{d\vec{R}}{dt}$: momento linear total do sistema (59.2)

↳ Eqs. (59.1) e (59.2) : $M \frac{d^2\vec{R}}{dt^2} = \frac{d\vec{P}}{dt} = \vec{F}$ (59.3)

Dessa forma:

· Eq. (59.1): movimento CM = mov. partícula massa M sob força externa $\vec{F}$;

· Eq. (59.2): momento linear total = momento partícula massa M c/ velocidade $V = V_{CM} = \frac{d\vec{R}}{dt}$;

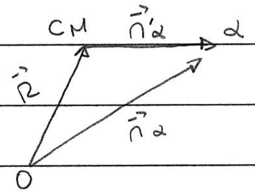
· Eqs. (59.2) e (59.3) : se $\vec{F} = 0$

↳ momento linear total $\vec{P}$ é conservado !

II) Conservação momento angular total.

considerar: $\vec{r}_\alpha = \vec{R} + \vec{r}'_\alpha$ (60.1)

$\vec{r}'_\alpha$ ↳ posição α w.r.t. CM



como: $\vec{\ell}_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha$: mom. angular α w.r.t. O

e $\dfrac{d\vec{r}_\alpha}{dt} = \dfrac{d\vec{R}}{dt} + \dfrac{d\vec{r}'_\alpha}{dt}$

$$\hookrightarrow \vec{L} = \sum_\alpha \vec{\ell}_\alpha = \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha = \sum_\alpha m_\alpha \left(\vec{r}_\alpha \times \frac{d\vec{r}_\alpha}{dt} \right)$$

$$= \sum_\alpha m_\alpha \left(\vec{R} + \vec{r}'_\alpha \right) \times \left(\frac{d\vec{R}}{dt} + \frac{d\vec{r}'_\alpha}{dt} \right)$$

$$= \sum_\alpha m_\alpha \left(\vec{R} \times \frac{d\vec{R}}{dt} + \vec{r}'_\alpha \times \frac{d\vec{r}'_\alpha}{dt} + \underbrace{\vec{r}'_\alpha \times \frac{d\vec{R}}{dt} + \vec{R} \times \frac{d\vec{r}'_\alpha}{dt}}_{(I)} \right) \quad (60.2)$$

notar: $(I) = \left(\sum_\alpha m_\alpha \vec{r}'_\alpha \right) \times \dfrac{d\vec{R}}{dt} + \vec{R} \times \dfrac{d}{dt} \left(\sum_\alpha m_\alpha \vec{r}'_\alpha \right)$

como: $\sum_\alpha m_\alpha \vec{r}'_\alpha \underset{\uparrow \text{Eq. (60.1)}}{=} \sum_\alpha m_\alpha (\vec{r}_\alpha - \vec{R}) = \underbrace{\sum_\alpha m_\alpha \vec{r}_\alpha}_{\text{Eq. (56.1)} : M\vec{R}} - \underbrace{\left(\sum_\alpha m_\alpha \right)}_{M} \vec{R} = 0$: (60.3)

: CM w.r.t. CM !

↳ $(I) = 0$

↳ Eq. (60.2): $\vec{L} = \vec{R} \times \underbrace{\left(\sum_\alpha m_\alpha\right)\frac{d\vec{R}}{dt}}_{M} + \sum_\alpha \vec{r}'_\alpha \times \underbrace{m_\alpha \frac{d\vec{r}'_\alpha}{dt}}_{\vec{p}'_\alpha\text{: momento linear } \alpha \text{ w.r.t. CM}}$

Eq. (59.2): $\vec{P}$

↳ $\vec{L} = \underbrace{\vec{R} \times \vec{P}}_{\text{mom. angular CM w.r.t. } \underline{O}} + \underbrace{\sum_\alpha \vec{r}'_\alpha \times \vec{p}'_\alpha}_{\text{mom. angular total w.r.t. CM}}$: momento angular total w.r.t. $\underline{O}$ (61.1)

· por outro lado, temos que:

$$\frac{d\vec{\ell}_\alpha}{dt} = \underbrace{\frac{d\vec{r}_\alpha}{dt} \times \vec{p}_\alpha}_{=0} + \vec{r}_\alpha \times \frac{d\vec{p}_\alpha}{dt} \underset{\text{Eq. (58.3)}}{=} \vec{r}_\alpha \times \vec{F}_\alpha^{(e)} + \vec{r}_\alpha \times \sum_{\beta \neq \alpha} \vec{f}_{\alpha\beta}$$

↳ $\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\sum_\alpha \vec{\ell}_\alpha\right) = \sum_\alpha \frac{d\vec{\ell}_\alpha}{dt} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha^{(e)} + \underbrace{\sum_{\alpha \neq \beta} \vec{r}_\alpha \times \vec{f}_{\alpha\beta}}_{(\text{II})}$

notar: $(\text{II}) = \frac{1}{2} \sum_{\alpha \neq \beta} \vec{r}_\alpha \times \vec{f}_{\alpha\beta} + \underbrace{\vec{r}_\alpha \times \vec{f}_{\alpha\beta}}$

$\alpha \leftrightarrow \beta$: $\vec{r}_\beta \times \vec{f}_{\beta\alpha} = -\vec{r}_\beta \times \vec{f}_{\alpha\beta}$

$= \frac{1}{2} \sum_{\alpha \neq \beta} (\vec{r}_\alpha - \vec{r}_\beta) \times \vec{f}_{\alpha\beta} = 0$

↳ forma "forte" Eq. (58.2)

↳ $\frac{d\vec{L}}{dt} = \sum_\alpha \underbrace{\vec{r}_\alpha \times \vec{F}_\alpha^{(e)}}_{\text{torque sob } \alpha \text{ devido forças externas w.r.t. } \underline{O}} = \sum_\alpha \vec{N}_\alpha^{(e)} = \vec{N}$: resultante torque devido forças externas w.r.t. $\underline{O}$ (61.2) notar Eq. (61.2): se $\vec{N} = 0$

↳ momento angular total $\vec{L}$ é conservado!

notar novamente (II):

$$\underbrace{\sum_\alpha \underbrace{\sum_\beta \vec{r}_\alpha \times \vec{f}_{\alpha\beta}}_{\text{torque sob } \underline{\alpha} \text{ devido forças internas}}}_{\text{torque total interno!}} = 0 \quad \text{p/ forças internas centrais} \tag{62.1}$$

III) Conservação energia total,

notar: $$W_{12} = \sum_\alpha \int_1^2 \vec{F}_\alpha \cdot d\vec{r}_\alpha \tag{62.2}$$: trabalho sistema configuração $\underline{1}$ p/ " $\underline{2}$!

$\vec{F}_\alpha$: força total sob partícula $\underline{\alpha}$: Eq. (57.5)

similar procedimento discutido no Cap. 2, Marion, verifica-se que Eq. (62.2) assume a forma:

$$W_{12} = \sum_\alpha \int_1^2 d\left(\frac{1}{2} m_\alpha v_\alpha^2\right) = T_2 - T_1 \tag{62.3}$$

onde: $T = \sum_\alpha T_\alpha = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2$: energia cinética total

como: $\vec{r}_\alpha = \vec{R} + \vec{r}\,'_\alpha$: Eq. (60.1)

↳ $\dfrac{d\vec{r}_\alpha}{dt} = \dfrac{d\vec{R}}{dt} + \dfrac{d\vec{r}\,'_\alpha}{dt}$

$$\hookrightarrow v_\alpha^2 = \left(\underbrace{\frac{d\vec{R}}{dt}}_{=\vec{V}_{CM}=\vec{V}} + \frac{d\vec{r}'_\alpha}{dt}\right)\cdot\left(\frac{d\vec{R}}{dt} + \underbrace{\frac{d\vec{r}'_\alpha}{dt}}_{=\vec{v}'_\alpha}\right) = V^2 + v'^2_\alpha + 2\,\vec{V}\cdot\frac{d\vec{r}'_\alpha}{dt}$$

$$\hookrightarrow T = \sum_\alpha T_\alpha = \frac{1}{2}\underbrace{\sum_\alpha m_\alpha}_{M} V^2 + \frac{1}{2}\sum_\alpha m_\alpha v'^2_\alpha + \vec{V}\cdot\frac{d}{dt}\underbrace{\left(\sum_\alpha m_\alpha \vec{r}'_\alpha\right)}_{=0\,:\ \text{Eq. }(60.3)}$$

$$\hookrightarrow T = \underbrace{\frac{1}{2}MV^2}_{\substack{\text{energia cinética}\\ \text{partícula massa } M\\ \text{e } \vec{V}=\vec{V}_{CM}=d\vec{R}/dt}} + \sum_\alpha \frac{1}{2}m_\alpha v'^2_\alpha \qquad (63.1)$$

velocidade partícula α w.r.t. CM

· por outro lado, Eq. (62.2) pode ser escrita como:

$$W_{12} = \sum_\alpha \int_1^2 \vec{F}_\alpha\cdot d\vec{r}_\alpha \underset{\text{Eq. }(57.5)}{=} \underbrace{\sum_\alpha \int_1^2 \vec{F}_\alpha^{(e)}\cdot d\vec{r}_\alpha}_{(I)} + \underbrace{\sum_{\alpha\neq\beta}\int_1^2 \vec{f}_{\alpha\beta}\cdot d\vec{r}_\alpha}_{(II)} \qquad (63.2)$$

se $\vec{F}_\alpha^{(e)}$ e $\vec{f}_{\alpha\beta}$: forças conservativas

$$\hookrightarrow \vec{F}_\alpha^{(e)} = -\vec{\nabla}_\alpha U(\vec{r}_\alpha) \quad \text{e} \quad \vec{f}_{\alpha\beta} = -\vec{\nabla}_\alpha \bar{U}(\vec{r}_\alpha,\vec{r}_\beta)$$

onde op. gradiente: (63.3)

$$\vec{\nabla}_\alpha = \sum_{i=1}^3 \hat{x}_{\alpha,i}\frac{\partial}{\partial x_{\alpha,i}} \quad ; \quad \vec{r}_\alpha = \sum_{i=1}^3 \hat{x}_{\alpha,i}\,x_{\alpha,i}$$

notar: $(I) = -\sum_\alpha \int_1^2 \left(\vec{\nabla}_\alpha U_\alpha(\vec{r}_\alpha)\right)\cdot d\vec{r}_\alpha = -\sum_\alpha U(\vec{r}_\alpha)\Big|_1^2$

como $\bar{U} = \bar{U}(\vec{r}_\alpha, \vec{r}_\beta)$, temos que:

$$d\bar{U} = \sum_{i=1}^{3} \frac{\partial \bar{U}}{\partial x_{\alpha,i}} dx_{\alpha,i} + \frac{\partial \bar{U}}{\partial x_{\beta,i}} dx_{\beta,i} = (\vec{\nabla}_\alpha \bar{U}) \cdot d\vec{r}_\alpha + (\vec{\nabla}_\beta \bar{U}) \cdot d\vec{r}_\beta$$

$$= - \vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha - \vec{f}_{\beta\alpha} \cdot d\vec{r}_\beta = - \vec{f}_{\alpha\beta} \cdot (d\vec{r}_\alpha - d\vec{r}_\beta)$$

$\hookrightarrow$ $$(\text{II}) = \frac{1}{2} \sum_{\alpha \neq \beta} \int_1^2 \vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha + \underbrace{\vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha}$$

$$+ \vec{f}_{\beta\alpha} \cdot d\vec{r}_\beta = - \vec{f}_{\alpha\beta} \cdot d\vec{r}_\beta$$

$$= \frac{1}{2} \sum_{\alpha \neq \beta} \int_1^2 \vec{f}_{\alpha\beta} \cdot (d\vec{r}_\alpha - d\vec{r}_\beta) = -\frac{1}{2} \sum_{\alpha \neq \beta} \int_1^2 d\bar{U}(\vec{r}_\alpha, \vec{r}_\beta)$$

$$= -\frac{1}{2} \sum_{\alpha \neq \beta} \bar{U}(\vec{r}_\alpha, \vec{r}_\beta) \Big|_1^2$$

$\hookrightarrow$ Eq. (63.2): $$W_{12} = - \sum_\alpha U(\vec{r}_\alpha) \Big|_1^2 - \frac{1}{2} \sum_{\alpha \neq \beta} \bar{U}(\vec{r}_\alpha, \vec{r}_\beta) \Big|_1^2$$

$$\equiv - (U_2 - U_1)$$

onde $U = \sum_\alpha U(\vec{r}_\alpha) + 1/2 \sum_{\alpha \neq \beta} \bar{U}(\vec{r}_\alpha, \vec{r}_\beta)$: energia potencial total (64.1)

$\hookrightarrow$ Eqs. (62.3) e (64.1):

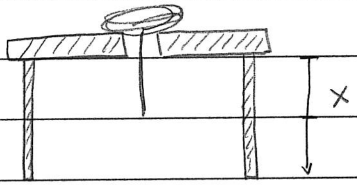
$W_{12} = T_2 - T_1 = U_1 - U_2 \quad \rightarrow \quad T_1 + U_1 = T_2 + U_2 = \text{cte}$ ∴ (64.2)

∴ energia total $E = T + V = \text{cte}$ p/ sistema conservativo

Ex.: P.9.15, Marion: considerar corda homogênea massa m, comprimento L;

em $t=0$, corda começa cair;

determinar $v=v(x)$; $a=a(t)$; energia $E=E(t)$



corda homogênea: densidade $\rho = m/L$

Eq. (57.4): $R = \frac{1}{M}\int_0^x dx' \rho x' = \frac{1}{2M}\rho x^2$

como $M = \rho x \to R = 1/2\, x$

Eqs. (59.2) e (59.3): $\dot{P} = \frac{d}{dt}(M\dot{R}) = Mg = \frac{d}{dt}\left(\rho x \cdot \frac{1}{2}\dot{x}\right)$

$\hookrightarrow \rho x g = \rho \dot{x} \cdot \frac{1}{2}\dot{x} + \rho x \cdot \frac{1}{2}\ddot{x} = \frac{1}{2}\rho(\dot{x}^2 + x\ddot{x})$

$\hookrightarrow 2xg = \dot{x}^2 + x\ddot{x} = v^2 + x\frac{dv}{dt}$

como $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = \frac{1}{2}\frac{dv^2}{dx}$

$\hookrightarrow v^2 + \frac{1}{2}x\frac{dv^2}{dx} = 2xg$

se $v^2 = ax^2 + bx + c \to \frac{dv^2}{dx} = 2ax + b$

$\hookrightarrow (ax^2 + bx + c) + (ax^2 + bx/2) = 2xg \to a = c = 0$ e $b = 4g/3$

$\hookrightarrow v = \sqrt{4gx/3}$

sobre a aceleração a:

Como $a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = \frac{1}{2}\frac{d v^2}{dx} = \frac{1}{2}\cdot\frac{4g}{3} = \frac{2}{3}g$

sobre a energia:

Eq. (63.1) : $T = \frac{1}{2} M V^2 = \frac{1}{2}\cdot \rho x \cdot \frac{4}{3} g x = \frac{2}{3}\rho g x^2$

Eq. (64.1) : $U = -MgR = -\rho x \cdot g \cdot \frac{1}{2} x = -\frac{1}{2}\rho g x^2$

↳ $E = T + U = \frac{1}{6}\rho g x^2 = \frac{1}{6} mg \frac{x^2}{L}$

· Colisões elásticas.

considerar: sistema $n=2$ partículas;
processo: espalhamento

ideia: estado inicial conhecido $\xrightarrow[\text{de conservação}]{\text{teoremas}}$ determinação estado final

ou descrição cinemática do processo de espalhamento;

Obs.:
(i) informação (detalhada) sobre interação entre partículas não é necessária, apenas a
hipótese: interação de curto alcance:

$$U(\vec{r}_1, \vec{r}_2) \to 0 \quad p/ \quad |\vec{r}_1 - \vec{r}_2| \gg a \qquad (66.1)$$

onde a: comprimento característico do potencial (sistema);

Eq. (66.1) → permite definição estados inicial e final:
partículas 1 e 2 livres p/ $|\vec{r}_1 - \vec{r}_2| \gg a$;

(ii) teoremas de conservação OK ~ interação entre partículas OK c/ 3ª lei de Newton.

· hipótese: colisões elásticas: conservação energia total das partículas 1 e 2 !

· notação: referencial laboratório: LAB
" centro de massa: CM

· m_i : massa partícula $\underline{i} = 1,2$

· $\vec{u}_i$: velocidade inicial partícula $\underline{i}$ w.r.t. LAB
$\vec{u}'_i$: " " " " " CM

· $\vec{v}_i$: " final " " " LAB
$\vec{v}'_i$: " " " " " CM

(67.1)

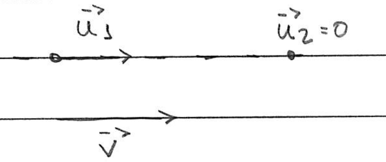
· $\vec{V} = \dfrac{d\vec{R}}{dt} = \vec{V}_{CM}$: velocidade do CM

· hipótese : $\vec{u}_2 = 0$
e sistema c/ simetria axial

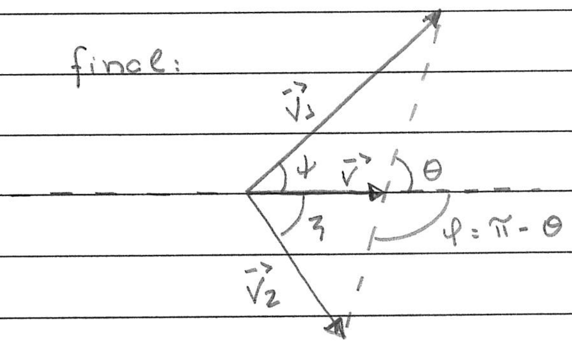
Geometria do processo:

estados inicial: final:

LAB:



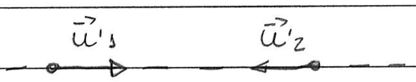
(a)



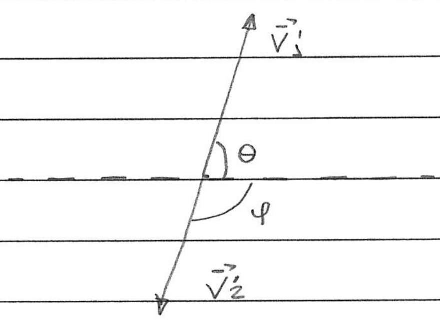
(b)

(67.2)

CM:



(c)



(d)

· REF CM:

(i) conservação momento linear total:

$$m_1\vec{u}'_1 + m_2\vec{u}'_2 = m_1\vec{v}'_1 + m_2\vec{v}'_2 = \frac{d}{dt}\underbrace{\left(m_1\vec{r}'_1 + m_2\vec{r}'_2\right)}_{=0\ :\ \text{Eq. }(60.3)} = 0$$

$\hookrightarrow$ $\vec{u}'_2 = \frac{m_1}{m_2}\vec{u}'_1$: $\vec{u}'_1 \parallel \vec{u}'_2$: Fig. (67.2)(c)

(68.1)

$\vec{v}'_2 = \frac{m_1}{m_2}\vec{v}'_1$: $\vec{v}'_1 \parallel \vec{v}'_2$: " " (d)

(ii) conservação energia total:

$$\frac{1}{2}m_1u'^2_1 + \underbrace{\frac{1}{2}m_2u'^2_2}_{\frac{1}{2}\frac{m_1^2}{m_2}u'^2_1} = \frac{1}{2}m_1v'^2_1 + \underbrace{\frac{1}{2}m_2v'^2_2}_{\frac{1}{2}\frac{m_1^2}{m_2}v'^2_1}\ :\ \text{Eq. }(68.1)$$

$\hookrightarrow$ $u'_1 = v'_1$ e $u'_2 = v'_2$ (68.2)

como $\vec{R} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2)$; $M = m_1 + m_2$: Eq. (56.1)

$\hookrightarrow$ $\vec{V} = \frac{d\vec{R}}{dt} = \frac{1}{M}(m_1\vec{u}_1 + m_2\vec{u}_2) = \frac{m_1}{M}\vec{u}_1$ (68.3)

hipotese $\vec{u}_2 = 0$

lembrar Eq. (60.1) : $\vec{r}_i = \vec{R} + \vec{r}'_i$

: ok c/ Figs. (67.2) (a) e (b)!

$\hookrightarrow$ $\vec{u}'_i = \vec{u}_i - \vec{V}$ e $\vec{v}'_i = \vec{v}_i - \vec{V}$ (68.4)

↳ $\vec{u}'_1 = \vec{u}_1 - \vec{V} = \left(1 - \frac{m_1}{M}\right)\vec{u}_1 = \frac{m_2}{M}\vec{u}_1$: $\vec{u}'_1 \parallel \vec{u}_1$:

: Fig. (67.2)(c)

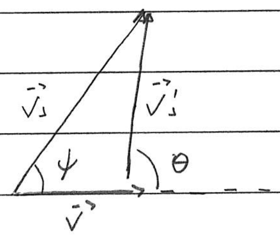
(69.1)

$\vec{u}'_2 = \vec{u}_2 - \vec{V} = -\vec{V} = -\frac{m_1}{M}\vec{u}_1$: $\vec{u}'_2 \propto -\vec{V}$: " " (d)

notar Eq. (68.4) : $\vec{v}'_1 = \vec{v}_1 - \vec{V}$:

: duas possibilidades:

· $V < v_1$:



(69.2)

· $V > v_1$: veja Fig. 9.11(b), Marion

· sobre os angulos de espalhamento ψ, ζ e θ:

notar Figs. (67.2):

$$v_1 \cos\psi = v'_1 \cos\theta + V$$
$$v_1 \sin\psi = v'_1 \sin\theta \qquad (69.3)$$

$$v_2 \cos\zeta = v'_2 \cos\varphi + V$$
$$v_2 \sin\zeta = v'_2 \sin\varphi \qquad (69.4)$$

como $v'_1 = u'_1 = \frac{m_2}{M}u_1$ e $v'_2 = u'_2 = \frac{m_1}{M}u_1$: Eq. (68.2)

Eq. (69.1)

temos que:

· $v_1 \cos\psi = \dfrac{m_2 u_1 \cos\theta}{M} + \dfrac{m_1 u_1}{M}$

$v_1 \sin\psi = \dfrac{m_2 u_1 \sin\theta}{M}$

↳ $\text{tg}\,\psi = \dfrac{\sin\theta}{\cos\theta + m_1/m_2}$ (70.1)

· $v_2 \cos\zeta = \dfrac{m_1 u_1 \cos\varphi}{M} + \dfrac{m_1 u_1}{M}$

$v_2 \sin\zeta = \dfrac{m_1 u_1 \sin\varphi}{M}$

↳ $\text{tg}\,\zeta = \dfrac{\sin\varphi}{1+\cos\varphi} \underset{\varphi = \pi - \theta}{=} \dfrac{\sin\theta}{1-\cos\theta} = \text{cotg}\,\dfrac{\theta}{2} = \text{tg}\left(\dfrac{\pi}{2} - \dfrac{\theta}{2}\right)$

(70.2)

↳ $2\zeta = \pi - \theta = \varphi$

notas:

Eq. (70.1) : relação entre ângulos de espalhamento ψ (REF. LAB) e θ (REF. CM) p/ partícula 1;

Eq. (70.2) : " " " " "
φ (REF. LAB) e θ (REF. CM) " " 2.

Eqs. (70.1) e (70.2) :

$\text{tg}\,\psi = \dfrac{\sin(\pi-\varphi)}{\cos(\pi-\varphi) + m_1/m_2} = \dfrac{\sin\varphi}{m_1/m_2 - \cos\varphi} = \dfrac{\sin 2\zeta}{m_1/m_2 - \cos 2\zeta}$;

(70.3)

: relação entre ângulos de espalhamento ψ (part. 1) e ζ (part. 2) no REF. LAB !

notar:

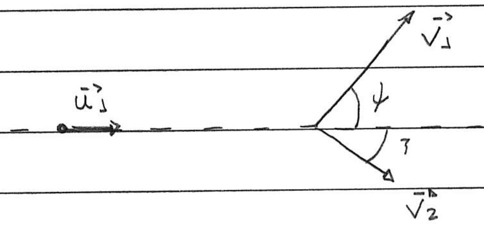
(i) se $m_1 = m_2 \rightarrow tg\psi = \dfrac{\sin\theta}{1+\cos\theta} = tg\dfrac{\theta}{2} \rightarrow 2\psi = \theta$

como $0 \leq \theta \leq \pi \rightarrow \psi_{MAX} = \pi/2$

e " $2\zeta = \pi - \theta \rightarrow 2\zeta = \pi - 2\psi \rightarrow \psi + \zeta = \pi/2$:

(71.1)

∴ e.g.:



· sobre a energia cinética:

notação:

$T_0 = \frac{1}{2} m_1 u_1^2$: energia cinética inicial w.r.t. LAB
(p/ $\vec{u}_2 = 0$)

$T_i = \frac{1}{2} m_i v_i^2$: " " final partícula i w.r.t. LAB

ideia: determinar relação entre T_0 e T_i;

temos que: $\dfrac{T_1}{T_0} = \dfrac{v_1^2}{u_1^2}$

Eq. (70.1): $\dfrac{v_1}{u_1} = \dfrac{m_2}{M} \cdot \dfrac{\sin\theta}{\sin\psi}$

(71.2)

e

$tg\psi = \dfrac{\sin\theta}{r + \cos\theta}$; $r \equiv m_1/m_2$

↳ $\sin\psi = \frac{1}{A}\sin\theta$ e $\cos\psi = \frac{1}{A}(n + \cos\theta)$

onde $A^2 = \sin^2\theta + (n + \cos\theta)^2 = 1 + n^2 + 2n\cos\theta$

↳ $\frac{T_1}{T_0} = \left(\frac{m_2}{M}\right)^2 \frac{\sin^2\theta}{\sin^2\psi} = \left(\frac{m_2}{M}\right)^2 \cdot \left(1 + n^2 + 2n\cos\theta\right)$

↳ $\frac{T_1}{T_0} = 1 - \frac{2m_1m_2}{M^2}(1 - \cos\theta)$

(72.1)

como $T_0 = T_1 + T_2 \rightarrow \frac{T_2}{T_0} = 1 - \frac{T_1}{T_0} = \frac{2m_1m_2}{M^2}(1 - \cos\theta)$

notar: se $m_1 = m_2 \rightarrow 2m_1m_2/M^2 = 1/2$

↳ $\frac{T_1}{T_0} = \frac{1}{2}(1 + \cos\theta) = \cos^2\theta/2 = \cos^2\psi$

(72.2)

$\frac{T_2}{T_0} = \frac{1}{2}(1 - \cos\theta) = \sin^2\theta/2 = \sin^2\psi$

Eq. (71.1)

p/ $m_1 \neq m_2$, verifica-se que T_1/T_0 em termos ψ assume a forma (veja pg. 72.1):

$$\frac{T_1}{T_0} = \left(\frac{m_1}{M}\right)^2 \left(\cos\psi \pm \underbrace{\left(\frac{m_2^2}{m_1^2} - \sin^2\psi\right)^{1/2}}_{(I)}\right)^2 \qquad (72.3)$$

notar: $(I) = 0 \rightarrow \sin^2\psi_{MAX} = \left(\frac{m_2}{m_1}\right)^2 \rightarrow \psi_{MAX} = \sin^{-1}\left(\frac{m_2}{m_1}\right)$

se $\psi > \psi_{MAX} \rightarrow T_1/T_0 \in \mathbb{C}$

(72.4)

↳ $\psi_{MÁX}$: angulo MAX espalhamento partícula 1

· Sobre a Eq. (72.3):

nesse caso, é interessante considerar o processo no REF. LAB;

conservação momento linear e energia:

$$m_1 u_1 = m_1 V_1 \cos\psi + m_2 V_2 \cos\xi \qquad : \text{Figs. (67.2) (a) e (b)}$$
$$0 = m_1 V_1 \sin\psi - m_2 V_2 \sin\xi \qquad (72.5)$$

$$\text{e} \quad \frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 \qquad (72.6)$$

notar Eq. (72.5): $(m_1 u_1 - m_1 V_1 \cos\psi)^2 = (m_2 V_2 \cos\xi)^2$ (+)
$$(m_1 V_1 \sin\psi)^2 = (m_2 V_2 \sin\xi)^2$$

$$\hookrightarrow (m_1 u_1)^2 + (m_1 V_1)^2 - 2 m_1^2 u_1 V_1 \cos\psi = \underbrace{(m_2 V_2)^2}_{m_2 m_1 (u_1^2 - V_1^2)}$$

$$* \, {}^1\!/m_1 m_2 : \quad (1+r) V_1^2 - 2 r u_1 V_1 \cos\psi + (r-1) u_1^2 = 0$$

$$\hookrightarrow \frac{V_1}{u_1} = \frac{1}{1+r} \left(r\cos\psi \pm \left(\underbrace{r^2 \cos^2\psi}_{\cancel{r^2} - r^2 \sin^2\psi} - (\cancel{r^2} - 1) \right)^{1/2} \right)$$

$$= \underbrace{\frac{r}{1+r}}_{m_1/(m_1+m_2)} \left(\cos\psi \pm \left(r^{-2} - \sin^2\psi \right)^{1/2} \right) \quad ; \quad r = m_1/m_2$$

$\hookrightarrow$ Eq. (72.3)

se $m_1 >> m_2 \rightarrow \psi_{MAX} = 0$

se $m_1 = m_2 \rightarrow \psi_{MAX} = \pi/2$

Ex.1: Exemplo 9.8, Marion: considerar espalhamento elástico
partícula massa m_1 e $\vec{u}_1 \neq 0$
e " " m_2 " $\vec{u}_2 = 0$;
determinar ângulo de espalhamento ψ (REF. LAB) se
$v_1 = \frac{2}{3} u_1$;

Eq. (72.3) : $\frac{T_1}{T_0} = \left(\frac{v_1}{u_1}\right)^2 = 1 - \underbrace{\frac{2 m_1 m_2}{M^2}}_{r/(1+r)^2} (1 - \cos\theta) = \frac{4}{9}$; $r = m_1/m_2$

$$\hookrightarrow \cos\theta = 1 - \frac{5}{18} \frac{(1+r)^2}{r} \equiv 1 - \alpha$$

$\hookrightarrow \sin^2\theta = 1 - (1-\alpha)^2 = 2\alpha - \alpha^2$

Eq. (71.2) : $tg\,\psi = \frac{\sin\theta}{r + \cos\theta} = \frac{(2\alpha - \alpha^2)^{1/2}}{r + 1 - \alpha}$:

: ângulo de espalhamento partícula 1 em termos
razão $r = m_1/m_2$!

notar: $\psi \in \mathbb{R} \rightarrow 2\alpha - \alpha^2 \geq 0 \rightarrow \alpha(2-\alpha) \geq 0 \rightarrow \alpha \leq 2$

$\hookrightarrow \frac{5}{18} \frac{(1+r)^2}{r} \leq 2$ ou $-5r^2 + 26r - 5 \geq 0$

$\hookrightarrow \frac{1}{5} \leq r \leq 5$ ou $\frac{1}{5} \leq \frac{m_1}{m_2} \leq 5$

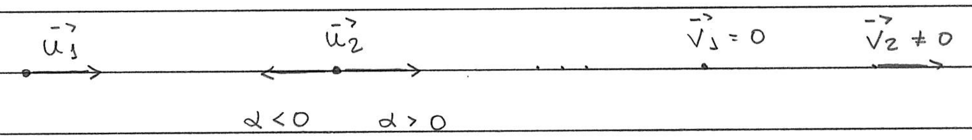
Ex. 2: P.9.36, Marion: considerar colisão elástica

partícula massa m_1 e $\vec{u}_1 \neq 0$

e " " m_2 e $\vec{u}_2 = \alpha \vec{u}_1$, $\alpha = cte$;

hipóteses: energias cinéticas iniciais $T_1 = T_2$ e $\vec{v}_1 = 0$;

determinar razões: m_1/m_2 e u_1/u_2;



: REF. LAB

conservação momento linear:

$$m_1 \vec{u}_1 + m_2 \vec{u}_2 = m_1 \vec{v}_1 + m_2 \vec{v}_2 \rightarrow m_1 \vec{u}_1 + \alpha m_2 \vec{u}_1 = m_2 \vec{v}_2$$

$$\hookrightarrow \vec{v}_2 = \frac{1}{m_2}(m_1 + \alpha m_2)\vec{u}_1 \rightarrow \vec{v}_2 \parallel \vec{u}_1$$

Como $T_{1I} = T_{2I} \rightarrow \frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_2 \alpha^2 u_1^2 \rightarrow m_1 = \alpha^2 m_2$

conservação energia:

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$\underbrace{\frac{1}{2} m_1 u_1^2}_{\frac{1}{2}\alpha^2 m_2 u_1^2} + \frac{1}{2}\alpha^2 m_2 u_1^2 = \frac{1}{2} m_2 \cdot \underbrace{\frac{1}{m_2^2}(m_1 + \alpha m_2)^2}_{(\alpha^2+\alpha)^2} u_1^2$$

$$\hookrightarrow 2\alpha^2 = (\alpha^2 + \alpha)^2 = \alpha^2 (1 + \alpha)^2$$

$$\hookrightarrow \alpha = 0 \text{ e } 1 + \alpha = \pm\sqrt{2} \rightarrow \alpha = -1 \pm \sqrt{2}$$

Obs.: as relações entre os ângulos de espalhamento ψ (REF. LAB) e θ (REF. CM), permitem determinar a relação entre a

$d\sigma(\psi)/d\Omega$: secção de choque diferencial w.r.t. LAB

e a $d\sigma(\theta)/d\Omega'$: " " " " " CM:

: Eq. (46.2); p/ detalhes, veja pg. 74.1.

· Colisões inelásticas.

caso geral, conservação de energia p/ processo colisão partículas m_1 e m_2, REF. LAB:

$$Q + \underbrace{\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2}_{T_{INICIAL}} = \underbrace{\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2}_{T_{FINAL}} \qquad (74.1)$$

se $Q = 0$ -> colisão elástica

$Q < 0$ -> " inelástica : $T_{FINAL} < T_{INICIAL}$

notar: Q: medida energia perdida no processo!

de fato, grau de inelasticidade do processo pode ser determinado via:

$$\varepsilon = \frac{|v_2 - v_1|}{|u_2 - u_1|} \quad : \text{coeficiente de restituição} \qquad (74.2)$$

c/ $0 \leq \varepsilon \leq 1$

· sobre a Eq. (46.2):

Fig. (69.2) ⊕ lei dos senos: $\dfrac{V}{\sin(\theta-\psi)} = \dfrac{V'_1}{\sin\psi}$

⊕ Eqs. (68.3) e (69.4): $\dfrac{m_1 u_1/M}{\sin(\theta-\psi)} = \dfrac{m_2 u_1/M}{\sin\psi}$

↳ $n = \dfrac{m_1}{m_2} = \dfrac{\sin(\theta-\psi)}{\sin\psi}$

↳ $n\sin\psi = \sin(\theta-\psi) \to \theta = \sin^{-1}(n\sin\psi) + \psi$; p/ $n<1$.

considerando $n = n(\theta,\psi)$, temos que:

$$dn = \frac{dn}{d\theta}d\theta + \frac{dn}{d\psi}d\psi$$

por outro lado, p/ razão $n = m_1/m_2$ fixa, temos $dn = 0$

↳ $\dfrac{d\theta}{d\psi} = -\dfrac{dn/d\psi}{dn/d\theta}$

Como $\dfrac{dn}{d\psi} = -\dfrac{\cos(\theta-\psi)}{\sin\psi} - \dfrac{\sin(\theta-\psi)\cos\psi}{\sin^2\psi}$

e $\dfrac{dn}{d\theta} = \dfrac{\cos(\theta-\psi)}{\sin\psi}$

↳ $\dfrac{d\theta}{d\psi} = 1 + \dfrac{\sin(\theta-\psi)\cos\psi}{\cos(\theta-\psi)\sin\psi} = \dfrac{\sin(\theta-\psi+\psi)}{\cos(\theta-\psi)\sin\psi}$

$\hookrightarrow \frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi} = \frac{1}{\cos(\theta-\phi)} \cdot \frac{\sin^2\theta}{\sin^2\phi}$

como $n\sin\phi = \sin(\theta-\phi) \to \cos(\theta-\phi) = \sqrt{1-n^2\sin^2\phi}$

notar:

$$n = \frac{\sin(\theta-\phi)}{\sin\phi}$$

$* \cos\phi \quad : \quad n\cos\phi = \frac{\sin(\theta-\phi)\cos\phi}{\sin\phi}$

$+ \cos(\theta-\phi) \quad : \quad n\cos\phi + \cos(\theta-\phi) = \underbrace{\frac{\sin(\theta-\phi)\cos\phi}{\sin\phi} + \cos(\theta-\phi)}_{(I)}$

como $(I) = \frac{\sin\theta\cos^2\phi}{\sin\phi} + \cos\theta\cos\phi - \cos\theta\cos\phi + \sin\theta\sin\phi$

$$= \frac{\sin\theta}{\sin\phi}\left(\cos^2\phi + \sin^2\phi\right) = \frac{\sin\theta}{\sin\phi}$$

$\hookrightarrow \frac{\sin\theta}{\sin\phi} = n\cos\phi + \cos(\theta-\phi) = n\cos + \sqrt{1-n^2\sin^2\phi}$

se $\varepsilon = 1$: colisão perfeitamente elástica

$\varepsilon = 0$: " " inelástica

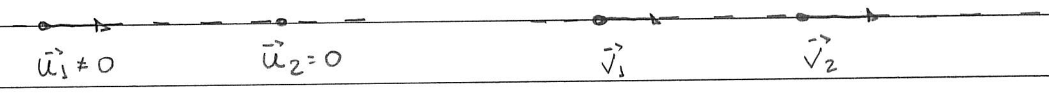
Obs.: Eq. (74.2) válida p/ colisões frontais;

p/ colisões oblíquas, Eq. (74.2) é válida p/ direção normal ao plano de contato : veja Fig. 9.18, Marion!

· consideramos: colisão frontal elástica, onde $\vec{u}_1 \neq 0$ e $\vec{u}_2 = 0$:

estado inicial　　　　　　　　estado final

REF. LAB:



(75.1)

conservação momento linear e energia:

$$m_1 u_1 = m_1 v_1 + m_2 v_2 \rightarrow v_2 = \frac{m_1}{m_2}(u_1 - v_1) \equiv r(u_1 - v_1)$$

$$\frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 \left(\frac{m_1}{m_2}\right)^2 \left(u_1^2 + v_1^2 - 2u_1 v_1\right)$$

$$\hookrightarrow (1+r) v_1^2 - 2u_1 r v_1 + (r-1) u_1^2 = 0$$

$$\hookrightarrow u_1 = v_1 \text{ (trivial) } \underline{\underline{e}} \; v_1 = \frac{r-1}{r+1} u_1$$

temos que : $v_2 - v_1 = r u_1 - (r+1) v_1 = r u_1 - (r-1) u_1 = u_1$

$\hookrightarrow \varepsilon = 1$, p/ $u_2 = 0$!

Ex. P.4.20, Symon: considerar colisão frontal inelástica caracterizada por um coef. de restituição ε, onde $\vec{u}_1 \neq 0$ e $\vec{u}_2 = 0$; determinar $-Q$!

similar Fig.(75.1), a conservação do momento linear

$\hookrightarrow V_2 = r(u_1 - v_1)$

nesse caso, a conservação da energia assume a forma:

$$\frac{1}{2} m_1 u_1^2 + Q = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 \quad : \text{REF. LAB}$$

$$= \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_1 \cdot r (u_1 - V_1)^2$$

$$\hookrightarrow Q = \frac{1}{2} m_1 (V_1^2 - u_1^2) + \frac{1}{2} m_1 \cdot r (u_1 - v_1)^2$$

como $\varepsilon u_1 = |V_2 - V_1| = |r u_1 - r V_1 - V_1| = |r u_1 - (r+1) V_1|$

se $r u_1 > (r+1) V_1 \rightarrow \varepsilon u_1 = r u_1 - (r+1) V_1 \rightarrow V_1 = \frac{r - \varepsilon}{r+1} u_1$:

: ok se $r > \varepsilon$!

$$\hookrightarrow Q = \frac{1}{2} m_1 \left((V_1 + u_1)(V_1 - u_1) + r(u_1 - V_1)^2 \right)$$

$$= \frac{1}{2} m_1 u_1^2 \cdot \frac{(\varepsilon^2 - 1)}{r+1} < 0 \quad \rightarrow \frac{-Q}{T_0} = \frac{1 - \varepsilon^2}{1 + m_1/m_2} \quad (76.1)$$

se $r u_1 < (r+1) V_1 \xrightarrow[\text{verificar!}]{} -Q/T_0$: Eq. (76.1)